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Mismatch and the optimal control of linear systems with series time delays.

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MISMATCH AND THE OPTIMAL CONTROL

OF LINEAR SYSTEMS WITH SERIES TIME DELAYS

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SUMMARY

A feature common to predictor control schemes for plants with time delays is the use of a plant model. Any difference between plant and model is referred to as mismatch. The presence of mismatch introduces extra parameters which in a parametric optimisation problem may be exploited to produce performance improvement. A study is undertaken of the mechanism and opportunities for such improvement.

When the cost functional is quadratic a matched predictor control scheme is known to be optimal. However, it is important to determine a relationship between mismatch and degradation in performance. This will indicate the amount by which plant and model may differ before the performance of the control scheme becomes unacceptable. With this aim, algebraic expressions are obtained for the mismatched input and output. An examination of these expressions shows how mismatch effects the stability and performance of the control scheme.

When the plant contains measurement delays, the optimal matched control scheme requires prior knowledge of an initial state. An alternative control scheme operating without this prior knowledge is suboptimal and may be improved by mismatch. The types of mismatch that could produce improvement are determined.

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CONTENTS

INTRODUCTION

Plant with series time delays, delays in either control or measurement, occur widely throughout industry. A control time delay occurs whenever a command is not implemented instantaniously. In an automated process this may be due to the time a moving part takes to achieve its correct alignment. Control time delays occur in Space applications as radio waves emitted from Earth take a non-negligible time to reach satellites. Another source of control time delays is the insensitivity of control mechanisms, which may prevent a control being implemented immediately a set point is reached (Oetker, 1963). An example of this is a thermostat, where the switching system does not respond immediately the preset temperature is reached (Choksy, 1962).

Implementing a feedback control scheme requires measurements be taken. A measurement delay occurs whenever a measurement cannot be made instantaniously. In the chemical industries measurement delays occur whenever the reactants flow through pipes connecting the reactor with the measuring device. If the required measurement is a chemical composition, a large time delay may result from any experimental analysis. In the rolling of steel (or like material) to uniform thickness, it is not possible to obtain accurate measurements at the roller because of subsequent cooling. As the equilibrium value of thickness is taken some distance from the roller, roller adjustments are subject to a measurement delay.

A common technique for analysing the behaviour of a control scheme is to obtain the roots of its characteristic equation in the Laplace domain. For a delay-free control scheme the characteristic equation is algebraic and locating the finite number of roots is straightforward. However, as the Laplace transform of a time delay is $e^{-s\tau}$ (τ is the length of the delay), the characteristic equation of a time delay control scheme is an exponential polynomial. The infinite number of roots of this transcendental equation are difficult to determine. Consequently, the behaviour of time delay schemes can be difficult to discern.

It is frequently the case that a time delay has a destabilising effect on a control scheme, restricting the values of loop gain and hence producing a sluggish response. Predictor control schemes may be used to control time delay systems, the name deriving from the use of a plant model as an element of prediction. In some cases, the predictive aspects of the model are seen to have the effect of removing the time delay from the closed loop. This externalising of the time delay reduces the design problem to that of a delay-free scheme, which allows the use of larger values of gain. However, any advantages of predictor control schemes have to be balanced against the fact that in most industrial applications, the plant will not be known exactly. This introduces the possibility of mismatch, that is, differences between plant and model. The aim of this thesis is to study the effects of mismatch on the stability and performance of predictor control schemes.

Chapter 1 of the thesis comprises a review of the literature on the effects of mismatch in predictor control schemes. This falls neatly into two categories, the beneficial effects of overestimating the time delay and the construction of mismatch bounds to maintain the stability of the control scheme. When the minimisation of a cost functional is a parametric optimisation problem, additional parameters introduced by the presence of mismatch may be exploited to produce performance improvement. In previous contributions on this topic, the improvement reported has been by overestimation of the time delay. Chapter 2 discusses the mechanism behind this improvement and develops a technique to estimate the optimal model delay. Motivated by this technique, three examples are presented which may be optimised by underestimating the plant delay. These examples show the potential for improvement by mismatch depends on the size of the plant delay. Therefore, the addition of time delays into predictor control schemes may allow further improvement in performance by mismatch. This is investigated with the results being applied to improve delay-free schemes.

In the remaining four chapters, a quadratic cost functional is associated with the predictor control schemes. The nature of this synthesis ensures that a matched control scheme is optimal. Chapters 3, 4 and 5 restrict attention to the case of control time delays, whereas Chapter 6 allows the presence of both control and measurement delays. Chapter 3 describes two optimal control schemes; the first is for a time-invariant problem with the second being its time-varying extension. The fact that the predictor control schemes of Chapter 3 incorporate plant models introduces the possibility of mismatch. A study of the effects of mismatch in the time-invariant case constitutes the material for the next two chapters. Chapter 4 contains the algebraic details which form the basis for the analytical and numerical results on stability and performance presented in Chapter 5. Chapter 6 allows measurement delays in the plant, in which case the optimal matched control scheme requires prior knowledge of an initial state. An alternative control scheme operating without this prior knowledge is suboptimal and may be improved by mismatch. The types of mismatch that could produce improvement are determined.

Chapter 1: A Review of Mismatch Problems

Introduction

A common method of controlling time delay plants is to implement a predictor control scheme, where a model of the plant provides the element of prediction. Any difference between plant and model is referred to as mismatch, the effects of which are examined in this thesis. This first chapter contains a review of the current mismatch literature, which centres around the Smith control scheme. An analysis of the Smith scheme is given in Section 1.1 and includes the motivation of an integral of squared error (ISE) cost functional. As the minimisation of this cost functional is a parametric optimisation problem, additional parameters introduced by mismatch may be exploited to produce performance improvement. Section 1.2 is an example of the Smith scheme where performance improvement is achieved by the over-estimation of plant delay.

The example of Section 1.2 is consistent with the general engineering practice of overestimating unknown time delays. Section 1.3 comprises reports of similar circumstances in which deliberately overestimating a time delay has been beneficial. However, an inappropriate choice of mismatch may be detrimental, producing an oscillatory response and in an extreme case an unstable control scheme. Section 1.4 reviews the construction of mismatch bounds within which the stability of a control scheme is guaranteed. The final section of Chapter 1, Section 1.5, contains references to the related topics of robustness and reduced-order modelling.

1.1: An Analysis of the Smith Control Scheme

The current mismatch literature is concentrated around the Smith control scheme, the power of which lies in the removal of the time delay from the feedback loop. The externalising of the delay reduces any design to that of a delay-free scheme, which allows higher gains and, in turn, produces a faster response. The following Laplace domain analysis of the Smith scheme commences by considering the delay-free control scheme of Figure 1.1.

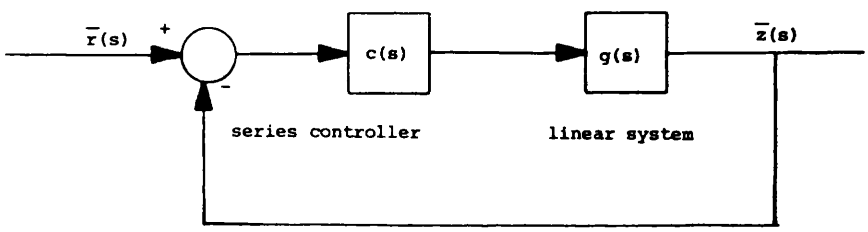


FIGURE 1.1: The Delay-Free Control Scheme

The control scheme of Figure 1.1 comprises a time-invariant, single-input/single-output linear system g(s), a series controller c(s) and unity negative feedback. It is assumed the control scheme is noise-free and the linear system has zero initial state. Any free parameters in c(s) are chosen to optimise the response according to some selected criteria. For this work, a compromise between the conflicting design requirements of fast response and small overshoot is achieved by choosing the cost of the delay-free control scheme to be the value of the ISE cost functional

$$J = \int_0^\infty \{r(t) - z(t)\}^2 \, dt \qquad (1.1)$$

In minimising this cost, the series controller is selected to force the output $z(t)$ to follow the input $r(t)$.

The work of Smith (1957, 1958, 1959) is concerned with the control of a plant with transfer function $g(s)e^{-s\tau}$, which consists of the linear system (subsequently referred to as the subplant) together with a delay τ in either control or measurement. It is assumed the initial function in the delay is the zero function, in other words, the output of the delay element is the zero function for a time equal to the length of the delay, followed by a delayed version of the input. It is also assumed that the connection between the delay and the subplant is inaccessible. Smith externalises the time delay by considering the desired response of the plant to be $z(t-\tau)$, a delayed version of the optimised delay-free response.

The choice of $c^*(s)$ necessary for the control scheme of Figure 1.2 to produce the desired output is determined by the following straightforward algebraic manipulation.

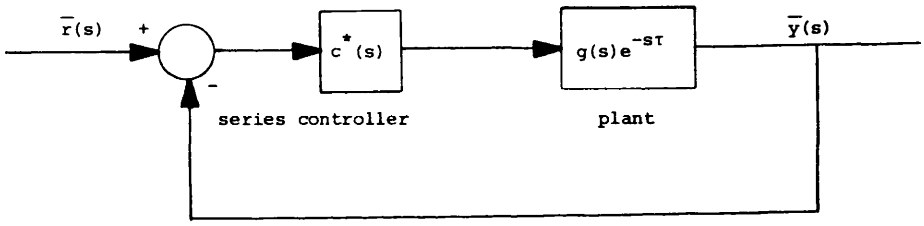


FIGURE 1.2: A Control Scheme for Plant with Series Delay

The Laplace transfer function of the control scheme of Figure 1.1 is

$$\frac{\bar{z}(s)}{\bar{r}(s)} = \frac{c(s)g(s)}{1+c(s)g(s)} \tag{1.2}$$

in which case, the desired transfer function for the control scheme of Figure 1.2 is

$$\frac{\bar{y}(s)}{\bar{r}(s)} = \frac{c(s)g(s)e^{-s\tau}}{1+c(s)g(s)} \tag{1.3}$$

In terms of $c^*(s)$, the control scheme of Figure 1.2 has transfer function

$$\frac{\bar{y}(s)}{\bar{r}(s)} = \frac{c^*(s)g(s)e^{-s\tau}}{1+c^*(s)g(s)e^{-s\tau}} \tag{1.4}$$

therefore, to produce the desired output, it is necessary to choose controller $c^*(s)$ to satisfy

$$\frac{c(s)g(s)e^{-s\tau}}{1+c(s)g(s)} = \frac{c^*(s)g(s)e^{-s\tau}}{1+c^*(s)g(s)e^{-s\tau}} \tag{1.5}$$

Rearranging (1.5) shows that the desired controller is given by

$$c^*(s) = \frac{c(s)}{1+c(s)g(s)(1-e^{-s\tau})} \tag{1.6}$$

Implementing this controller, Figure 1.2 takes the form of Figure 1.3.

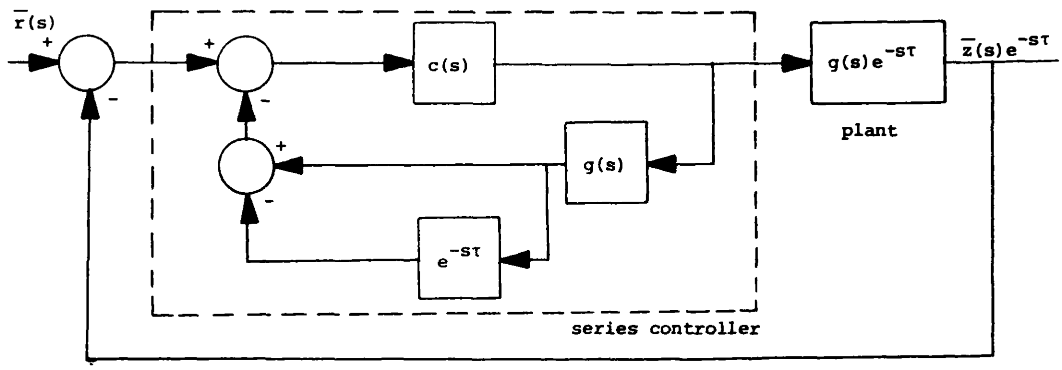


FIGURE 1.3: The Smith Control Scheme

It is observed that the series controller incorporates models of both the delay-free subplant and the series time delay. At this stage of the analysis it is assumed that the plant and model are identical, that is, the control scheme is matched. The alternative realisation of Figure 1.4, favoured in this work, highlights the important role of the plant model, which is constructed to allow an accessible delay-free output.

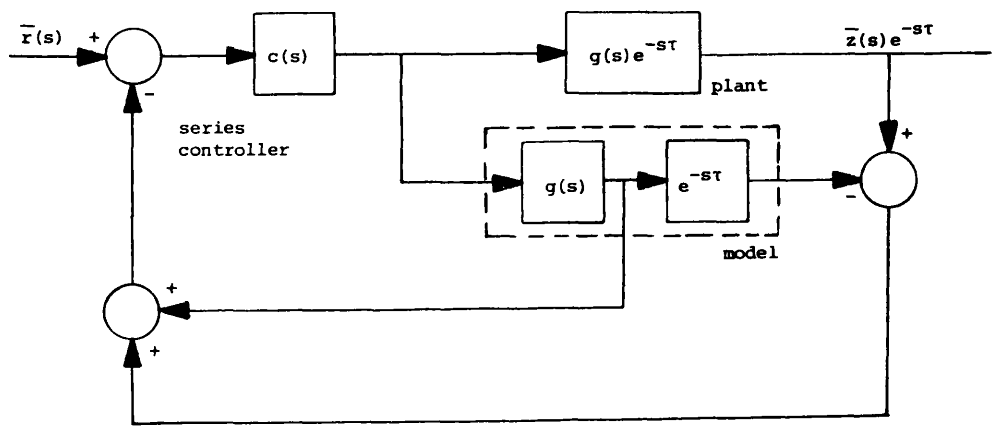


FIGURE 1.4: The Smith Scheme, an Alternative Realisation

The delay-free output of the model is an advanced version of the plant output. The fact that the model predicts the plant output is the basis of predictor control schemes and explains why the Smith scheme is often referred to as the Smith predictor.

As the output $z(t-\tau)$ follows $r(t-\tau)$, a sensible choice for the cost of the matched Smith scheme is

$$J = \int_{\tau}^{\infty} \{r(t-\tau)-z(t-\tau)\}^2 \, h(t-\tau)dt \tag{1.7}$$

where $h(t)$ is the Heaviside step function. With a change of variable cost (1.7) is seen to equal the delay-free cost (1.1). The power of the matched Smith scheme is seen from (1.3) which shows its characteristic equation

$$1 + c(s)g(s) = 0 \tag{1.8}$$

is that of the delay-free scheme, having a finite rather than an infinite number of roots. When compared with conventional feedback schemes for time delay plants, the higher gains available to the matched Smith scheme give it a superior performance (Nielsen, 1969; Ross, 1977). The Smith scheme has been extended by Alevisakis and Seborg (1973) to multivariable systems with single time delay, and by Ogunnaike and Ray (1979) to multivariable systems with multiple time delays.

In the past, the expense of hardware necessary for the plant model prevented the Smith scheme finding wide industrial application. However, the advent of microprocessor technology has made the industrial implementation of the Smith scheme more attractive (Alevisakis and Seborg, 1974; Prasad and Krishnasway, 1975; Byron, Cox and Ball, 1979). In particular, pure time delays may now be realised by digital storage techniques. A major problem of any predictor control scheme is that in most practical situations the plant will not be known accurately.

This introduces the possibility of mismatch, that is, differences between plant and model. In this work, it is assumed that the model is a linear system $g_o(s)$ of the correct order, with a series delay τ_o, but that precise parameter values may be unknown. The mismatched Smith control scheme is shown in Figure 1.5.

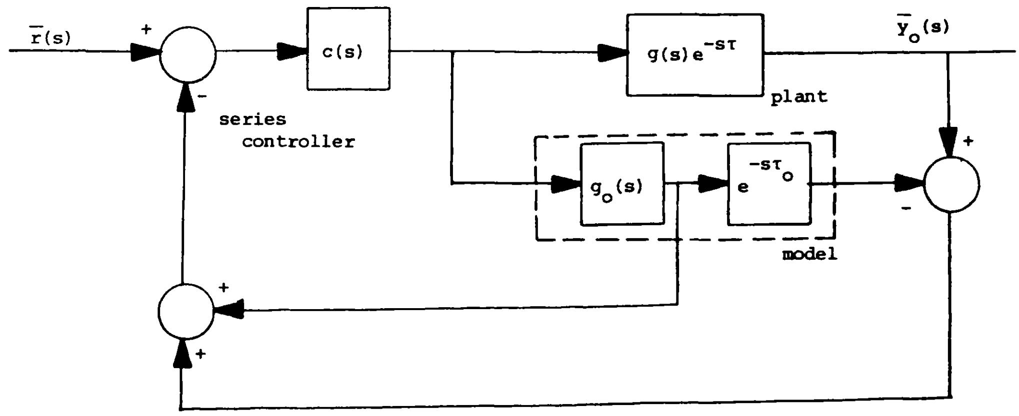


FIGURE 1.5: The Mismatched Smith Control Scheme

The cost of the mismatched Smith scheme is constructed by substituting the matched with the mismatched output

$$J_o = \int_{\tau}^{\infty} \{r(t-\tau) - y_o(t)\}^2 \, h(t-\tau)dt \tag{1.9a}$$

The major parts of Chapters 1 and 2 are concerned with mismatch in delay, for which

$$J_\tau(\tau_o) = \int_{\tau}^{\infty} \{r(t-\tau) - y_{\tau,\tau_o}(t)\}^2 h(t-\tau)dt \tag{1.9b}$$

is the more detailed notation adopted for cost.

The transfer function of the mismatched Smith scheme

$$\frac{\bar{y}_o(s)}{\bar{r}(s)} = \frac{c(s)g(s)e^{-s\tau}}{1+c(s)g_o(s)+c(s)(g(s)e^{-s\tau}-g_o(s)e^{-s\tau_o})} \tag{1.10}$$

reduces to (1.3), the transfer function of the control schemes of Figures 1.2, 1.3 and 1.4, when plant and model are identical. The infinite number of roots of the transcendental characteristic equation

$$1 + c(s)g_o(s) + c(s)(g(s)e^{-s\tau} - g_o(s)e^{-s\tau_o}) = 0 \tag{1.11}$$

are difficult to determine (Manitus and Olbrot, 1979), consequently, the behaviour of the mismatched Smith scheme is difficult to discern. Where the presence of mismatch is understood, the mismatched Smith scheme will subsequently be referred to simply as the Smith scheme.

The earliest authors to address the problems created by mismatch include Buckley (1963), Wheater (1966), Gray and Hunt (1971) and Marshall (1971, 1974). They used simulation techniques to examine the effects of mismatch in gain, time constant and time delay. These early contributions conclude that the Smith scheme is most sensitive to mismatch in delay. Wheater (1966) remarks that the problems of parameter identification must be included in any assessment of the importance of parameter sensitivity. The works of Gray and Hunt (1971) and Marshall (1971, 1974) use the Smith scheme in a digital control setting. Buckley (1963) is the first of many authors to state that the response of the Smith scheme is less damped with underestimation of delay than with overestimation. Section 1.3 comprises a collection of similar instances where deliberately overestimating a time delay has been beneficial; Section 1.2 considers one of these instances in detail.

Compared with matched cost functional (1.7), cost functionals (1.9) contain extra parameters owing to the presence of mismatch. This introduces the possibility of the cost of the mismatched Smith scheme being less than the corresponding matched cost. An example of improvement in performance by mismatch is now presented; it is developed from an example in Marshall and Salehi (1982).

1.2 An Example of Improvement in Performance by Mismatch

The first part of the example is a delay-free optimisation involving the selection of a series controller. The analysis of Section 1.1 shows this determines the cost of the matched Smith scheme. An investigation is then undertaken into how the cost of the Smith scheme varies with model delay.

The delay-free control scheme of Figure 1.1 is assumed to incorporate the linear subplant

$$g(s) = \frac{1}{s(s+1)} \tag{1.12a}$$

and a constant series controller

$$c(s) = k, \quad 0 < k \leq 1 \tag{1.12b}$$

The upper bound is imposed on the controller, to ensure the example makes sense from a physical point of view. The characteristic equation of the control scheme

$$s^2 + s + k = 0 \tag{1.12c}$$

has both roots in the left-half plane, in which case, the control scheme is stable and it can be shown analytically that cost functional (1.1) takes the form

$$J = \frac{k + 1}{2k} \tag{1.12d}$$

Clearly, (1.12d) is minimised by the maximum value of k, $k = 1$, for which the delay-free optimisation is given by $J = 1$.

Subplant (1.12a) is now considered to be part of a plant with delay $\tau = 1$ in either control or measurement. This plant is incorporated into the Smith scheme together with the series controller $c(s) = k = 1$. For a unit step input, the variation with model delay of the cost of this Smith scheme is shown in Figure 1.6

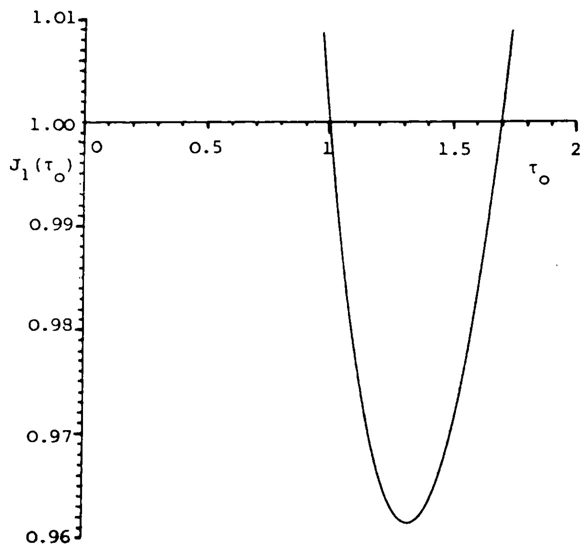


FIGURE 1.6: Variation of Cost with Model Delay

The cost of the matched Smith scheme is seen to equal the unity delay-free cost, as the horizontal axis is crossed by the curve when $\tau_o = \tau = 1$. Furthermore, decreasing the model delay from the matched value produces an increase in the cost of the Smith scheme. However, increasing the model delay above the matched value produces an initial decrease in cost which continues until the optimal model delay $(\tau_o)_{opt} = 1.32$ is reached, for which the cost of the Smith scheme is minimised. Finally, the cost of the Smith scheme is strictly increasing with model delays greater than $(\tau_o)_{opt}$. The optimal model delay is a 32% overestimate of the plant delay, for which $J_1(1.32) = 0.9614$, is a 3.86% improvement over the matched cost. This apparently small reduction actually corresponds to a significant improvement in the step response, as shown in Figure 1.7.

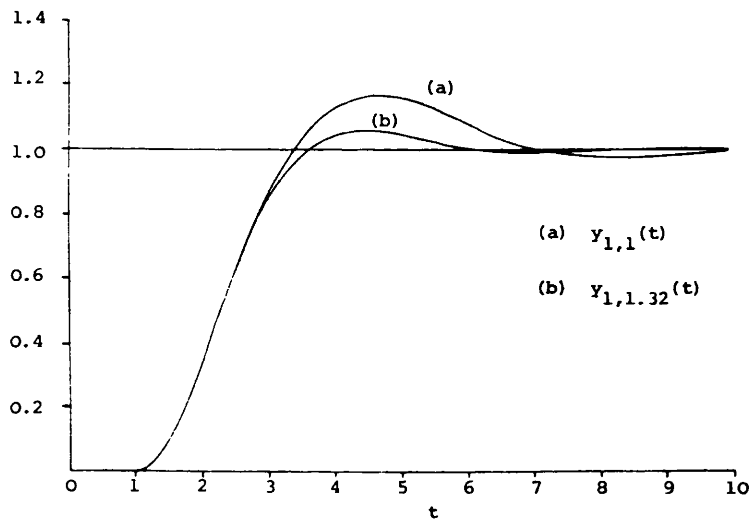


FIGURE 1.7: Improvement in Step Response by Mismatch

When the Smith scheme incorporates the optimal model delay the overshoot of the step response is 6.4%. This compares with 16.6% when the Smith scheme is matched, an improvement of 10.2%. The settling time has also been reduced by this astute introduction of mismatch. The method of improvement is consistent with the general engineering practice of overestimating unknown time delays. The next section comprises similar reports of circumstances where overestimation of a time delay has proved preferable to underestimation.

1.3 The Beneficial Effects of Overestimating the Time Delay

The experimental work of Alevisakis and Seborg (1974) considers the application of their multivariable Smith scheme to a pilot scale double effect evaporator. Despite inaccurate estimates of the time delays, in a comparison with conventional feedback control schemes, the Smith scheme allows the use of higher gains. Of particular interest is their result that the underestimation of time delays produces the more oscillatory responses.

The strategy of overestimating unknown time delays receives further support from Byron, Cox and Ball (1979), in a report concerning the control of the moisture content of sinter about a set point. (Sinter is a hard porous mass formed by combining fine mineral particles with coke and limestone. It forms the ferrous burden in the Blast Furnace.) The plant operating requirements are small settling time with little or no overshoot. The principle feature of the plant is a long delay which has a destabilising effect on closed loop performance. Employing the inner feedback loop of the Smith scheme allows the use of higher gains but introduces the possibility of mismatch. The variation of performance with mismatch in delay is detailed in Table 1.1 below:

Time Delay Error		Overshoot	Settling Time (seconds) 10%	Settling Time (seconds) 5%
Underestimation	10%	18%	880	1080
	5%	10%	790	900
Matched	0%	0%	500	550
Overestimation	5%	0%	720	800
	10%	0%	770	940

TABLE 1.1: Control of Moisture Content, Variation with Mismatch

The most immediate result is that whilst underestimation of delay increases overshoot, overestimation of delay produces no overshoot. Furthermore, settling time is more seriously impaired by the underestimation of time delay. In this situation, if a perfect model of the delay is not available, it is clearly better to overestimate the time delay. It is subsequently observed that these results are consistent with relevant results of Chapter 5.

Vit (1979) observes substantial improvement in the dynamic behaviour of an example of the Smith scheme, as a result of a large overestimate of the plant delay. The Smith scheme considered incorporates a plant comprising the third-order subplant

$$g(s) = \frac{1}{s(s+1)^2} \tag{1.13}$$

and the series delay $\tau = 0.2$ seconds. Vit (1979) constructs a controllability index for the Smith scheme from the critical gain and the critical frequency of the scheme. The critical gain, k_c, is the value of gain k at which the control scheme output oscillates with constant amplitude. Moreover, the critical frequency, w_c, is the frequency of this constant amplitude output. Vit (1979) considers it desirable to have both these parameters as large as possible, and consequently defines a controllability index as their product. When this controllability index is maximised over all possible model delays, the optimal model delay is found to be approximately four times the plant delay. It is also noted that any underestimate of the plant delay impairs the performance of the control scheme. The numerical details are given in Table 1.2.

Model Value	w_c (rad/sec)	k_c	Controllability Index
0 (Delay-Free)	0.84	1.43	1.2012
0.2 (Matched)	1.00	2.00	2.0000
0.81 (Optimal)	1.48	2.75	4.0700

TABLE 1.2: Improvement in Controllability Index by Mismatch

One of the most recent papers to investigate the performance improvement by mismatch is that of Marshall and Salehi (1982), which continues the earlier explorations into the question of mismatch by Garland (1974, 1978) and Garland and Marshall (1974, 1975, 1978). In addition to the example considered in Section 1.2, Marshall and Salehi (1982) discuss a mechanism for improvement by mismatch. They show the mismatched error is an infinite series of terms the first of which is the matched error. Improvement by mismatch occurs when an appropriate choice of model delay allows the first term to be reduced by subsequent terms. This makes the mismatched error "nearer" the zero function, which ensures the mismatched Smith scheme incurs a smaller cost. The optimal model delay may be found numerically, however, optimisation routines for time delay schemes are expensive in terms of computer time. Motivated by this fact, Marshall and Salehi (1982) develop an inexpensive technique to estimate the optimal model delay, based on the simulation of two delay-free curves. The estimation technique is seen to work well for the example of Section 1.2 where improvement is by overestimation of the plant delay.

The paper of Marshall and Salehi (1982) leaves several open questions, in particular; what is the basis of the general engineering practice of overestimating unknown time delays, and are there any examples of the Smith scheme which are improved by an underestimate of the plant delay? These question are taken up in Hocken, Salehi and Marshall (1983) where three examples of the Smith scheme are optimised by the underestimation of plant delay. The authors conjecture that improvement is by underestimation or overestimation depending on whether mismatch is being utilised to cancel a peak or a trough of the matched error. It is argued that this conjecture is consistent with the current literature which invariably suggests the overestimation of plant delay. The paper of Hocken, Salehi and Marshall (1983) is the basis of Chapter 2.

1.4 Mismatch and the Stability of Predictor Control Schemes

Section 1.3 contains several reports of underestimates of plant delay producing oscillatory responses. This suggests that if mismatch is severe enough the Smith scheme may become unstable. Section 1.4 reviews the construction of mismatch bounds within which the stability of a control scheme is guaranteed.

The paper of Ioannides, Rogers and Latham (1979) is primarily concerned with determining ranges of mismatch for which an example of the Smith scheme is stable. The example considered incorporates a stable second-order subplant. Simulation techniques produce stability regions for the simultanious mismatch of gain, time constant and time delay. The stability regions show that models maintaining stability have small time constants, small gains and large time delays. Ioannides, Rogers and Latham (1979) also assign an integral of time multiplied by absolute error (ITAE) cost functional to the Smith scheme. As with the ISE cost functional, it is minimised by an overestimate of the plant delay.

The problem of mismatch bounds is examined in a general Banach space setting by Owens and Raya (1982). The mismatch restrictions determined by a contraction mapping theorem in the frequency domain, are sufficient conditions for stability of the Smith scheme. These conditions are modified to allow variations in the plant parameters, thus obtaining bounds for a "double" mismatch problem. For multivariable linear systems with measurement delay, the results may yield a Nyquist structure. Other contributors to the frequency domain analysis of mismatch bounds are Palmor and Shinnar (1978), Palmor (1980, 1982) and Kantor and Andres (1980). Kantor and Andres (1980) consider an extension of the Smith scheme to accommodate plants $g_1(s)g_2(s)$ where the connection between $g_1(s)$ and $g_2(s)$ is inaccessible. However, a drawback of all the results obtained for this topic is the conservative nature of the mismatch constraints.

Following the work of Fuller (1968) and Mee (1973), Marshall, Ireland and Garland (1977) present a control scheme to minimise a quadratic cost functional for plants with delay in control. Chapter 3 contains an analysis of the control scheme together with a time-varying extension. The structure of the control schemes is similar to that of the Smith scheme, containing a plant model for purposes of prediction. However, the nature of the cost functional ensures the matched control scheme is optimal. A first brief study into how mismatch affects the control scheme was undertaken by Chotai (1980, 1981) using analogue simulation. A mathematical analysis of the mismatched control scheme is contained in Hocken and Marshall (1983).

Hocken and Marshall (1983) derive matrix integral equations satisfied by the mismatched input and output, which are straightforward to solve when the subplant is first-order. An analysis of the mismatched inputs and outputs provides mismatch bounds within which the control scheme is stable, together with a relationship between mismatch and degradation in performance. A more detailed analysis of the mismatched control scheme constitutes Chapters 4 and 5. Complementary aspects of Hocken and Marshall (1983) and Marshall and Salehi (1982) are summarised in Hocken, Marshall and Salehi (1983). Measurement delays are included in the analysis in Hocken and Marshall (1982) which forms the basis of Chapter 6.

1.5 Related Mismatch Studies

Robustness and reduced-order modelling are research topics closely related to the problems of mismatch. A control scheme is robust if its performance is insensitive to plant parameter variations. A collection of papers on robustness is included in Bell, Cook and Munro (1982). Other recent contributions in the area are from Åström (1980 a,b), Owens and Chotai (1982 a,b,c), Francis (1980) and Yahagi (1977). In practical situations the known plant may be so complex that controller design is based on a reduced-order model. A literature survey on this topic was undertaken by Genesio and Milanese (1976). Owens and Raya (1982) allow plant and model to be of different order by assuming they are related by a feedback operator. Reviews of recent research into mismatch and related subjects can be found in Garland and Marshall (1979) and Marshall (1979a, 1979b, 1981, 1982).

Conclusions

The Smith control scheme is designed to produce a delayed version of an optimised delay-free response. In so doing, the Smith scheme eliminates the time delay from the closed loop characteristic equation. However, in most practical applications of the Smith scheme the plant will not be known exactly and mismatch will result. The characteristic equation of the mismatched Smith scheme is an exponential polynomial, the infinite number of roots of which are difficult to determine.

An ISE cost functional is assigned to the mismatched Smith scheme. The extra parameters made available by the presence of mismatch introduce the possibility of the mismatched cost being less than the corresponding matched cost. An example is presented where a 3.86% improvement in performance is achieved by an overestimation of plant delay. Furthermore, this astute introduction of mismatch significantly reduces the overshoot and settling time of the step response.

The method of improvement is consistent with the general engineering practice of overestimating unknown time delays. Further examples of the benefits of overestimation appear in Buckley (1963), Alevisakis and Seborg (1974), Byron, Cox and Ball (1979) and Vit (1979). Marshall and Salehi (1982) discuss a mechanism for improvement by mismatch and develop an inexpensive technique for estimating the optimal model delay. However, the question as to the basis of the beneficial effects produced by overestimation is left open. This question is taken up in Hocken, Salehi and Marshall (1983) which constitutes Chapter 2.

The reports that underestimation of plant delay produce an oscillatory response suggests that if mismatch is severe enough the control scheme may become unstable. Mismatch bounds within which the stability of the Smith scheme is guaranteed are derived by Ioannides, Rogers and Latham (1979), Kantor and Andres (1980), Palmor (1980, 1982) and Owens and Raya (1982). Hocken and Marshall (1982, 1983) concentrate on a predictor control scheme which minimises a quadratic cost functional. Expressions for the mismatched input and output are obtained in the time domain. An analysis of these expressions provides the required mismatch bounds together with a relationship between mismatch and degradation in performance. A full account of this work is contained in the later chapters of this thesis. The final section of the mismatch review contains references to the related topics of robustness and reduced-order modelling.

Chapter 2: Improvement in Performance by Mismatch

Introduction

One of the most recent papers to investigate performance improvement by mismatch is that of Marshall and Salehi (1982). They discuss the mechanism of improvement and develop an inexpensive technique to estimate the optimal model delay. A thorough account of the mechanism of improvement and the estimation technique are given in Section 2.1. However, there remains several open questions, in particular; why is the overestimation of plant delay often beneficial, and are there any examples of the Smith scheme which are improved by an underestimate of the plant delay? Chapter 2 considers these questions, restricting attention to mismatch in delay.

A close examination of the estimation technique gives insight into why improvement has previously been by overestimation of plant delay. This leads to the examples of Section 2.2 which are optimised by the underestimation of plant delay. A first-order example yields an improvement in excess of thirty per cent for an underestimate of the plant delay. It is shown that a second-order example may also be improved by underestimation of plant delay, but the most significant improvement would appear to be achieved by overestimation. Finally, a third-order example is improved by either underestimation or over-estimation depending on the size of the plant delay.

The examples of Section 2.2 show that the potential for improvement by mismatch depends on the size of the plant delay. Therefore, the addition of time delays into the Smith scheme may allow further improvement in performance by mismatch. The augmented Smith scheme presented in Section 2.3 is a consequence of this idea. Finally, utilising results for time delay schemes, Section 2.4 shows the performance of delay-free schemes can be improved by the addition of an outer feedback loop containing two time delays. The contents of Chapter 2 appear in shortened form in Hocken, Salehi and Marshall (1983).

2.1: Estimating the Optimal Model Delay

The investigation into the mechanism of improvement commences with an expansion of the transfer function for the delay mismatched Smith scheme. The transfer function of the delay mismatched Smith scheme is given by

$$\frac{\bar{y}_{\tau,\tau_o}(s)}{\bar{r}(s)} = \frac{p(s)e^{-s\tau}}{1-p(s)(e^{-s\tau_o}-e^{-s\tau})} \tag{2.1}$$

where p(s) is the transfer function of the delay-free control scheme of Figure 1.1

$$p(s) = \frac{c(s)g(s)}{1+c(s)g(s)} \tag{2.2}$$

For $s \in \mathbb{C}$ with real part large enough

$$|p(s)(e^{-s\tau_o}-e^{-s\tau})| < 1 \tag{2.3}$$

and (2.1) can be expressed as a geometric series

$$\frac{\bar{y}_{\tau,\tau_o}(s)}{\bar{r}(s)} = p(s)e^{-s\tau}\left\{\sum_{n=1}^{\infty} p^{n-1}(s)(e^{-s\tau_o}-e^{-s\tau})^{n-1}\right\}$$

$$= \sum_{n=1}^{\infty} p^{n}(s)(e^{-s\tau_o}-e^{-s\tau})^{n-1}e^{-s\tau}$$

$$= p(s)e^{-s\tau} + \sum_{n=2}^{\infty} p^{n}(s)(e^{-s\tau_o}-e^{-s\tau})^{n-1}e^{-s\tau} \tag{2.4}$$

The mechanism of improvement is revealed by the form of the mismatched error

$$e_{\tau,\tau_o}(t) = r(t-\tau) - y_{\tau,\tau_o}(t) \tag{2.5}$$

which has Laplace transform

$$\bar{e}_{\tau,\tau_o}(s) = \bar{r}(s)e^{-s\tau} - \bar{y}_{\tau,\tau_o}(s) \tag{2.6}$$

Substituting (2.4) into (2.6)

$$\bar{e}_{\tau,\tau_o}(s) = \bar{r}(s)e^{-s\tau}(1-p(s)) - \sum_{n=2}^{\infty} p^n(s)(e^{-s\tau_o}-e^{-s\tau})^{n-1} e^{-s\tau}\bar{r}(s) \tag{2.7}$$

When the Smith scheme is matched, the transfer function is given by

$$\frac{\bar{y}_{\tau,\tau}(s)}{\bar{r}(s)} = p(s)e^{-s\tau} \tag{2.8}$$

and the Laplace transform of error by

$$\bar{e}_{\tau,\tau}(s) = \bar{r}(s)\ e^{-s\tau}(1-p(s)) \tag{2.9}$$

Clearly (2.9) is the first term of (2.7) in which case

$$\bar{e}_{\tau,\tau_o}(s) = \bar{e}_{\tau,\tau}(s) - \sum_{n=2}^{\infty} p^n(s)(e^{-s\tau_o}-e^{-s\tau})^{n-1} e^{-s\tau}\bar{r}(s) \tag{2.10}$$

The mechanism of improvement can be seen by taking inverse Laplace transforms throughout (2.10). An appropriate choice of model delay allows the first term of the mismatched error, namely the matched error, to be reduced by subsequent terms. This makes the mismatched error "nearer" the zero function which ensures the mismatched Smith scheme incurs a smaller cost. However, a note of caution, an inappropriate choice of model delay will cause the terms of the mismatched error to have a cummulative effect, producing a much poorer performance than in the matched case.

The optimal model delay can be found numerically, however, optimisation routines for time delay systems are expensive in terms of computer time. Consequently, an inexpensive technique of estimating the optimal model delay is desirable. The aim of the following discussion is to produce such a technique.

Expressing the model delay in terms of the plant delay

$$\tau_o = \tau + \varepsilon, \quad \varepsilon \geq -\tau \tag{2.11}$$

(2.10) may be rewritten in the form

$$\bar{e}_{\tau,\tau_o}(s) = \bar{e}_{\tau,\tau}(s) + \sum_{n=2}^{\infty} p^n(s)(e^{-s\tau} - e^{-s(\tau+\varepsilon)})^{n-1} e^{-s\tau}\bar{r}(s)$$

$$= \bar{e}_{\tau,\tau}(s) + \sum_{n=2}^{\infty} p^n(s)(1 - e^{-s\varepsilon})^{n-1} e^{-sn\tau}\bar{r}(s)$$

$$= \bar{e}_{\tau,\tau}(s) + \sum_{n=2}^{\infty} p^n(s)(s\varepsilon - \frac{s^2\varepsilon^2}{2!} + \frac{s^3\varepsilon^3}{3!} - \ldots)^{n-1} e^{-sn\tau}\bar{r}(s)$$

$$= \bar{e}_{\tau,\tau}(s) + p^2(s)s\varepsilon\, e^{-2s\tau}\bar{r}(s) + O(\varepsilon^2) \tag{2.12}$$

Assuming mismatch is small enough to neglect the terms comprising $O(\varepsilon^2)$, for a unit step input, the Laplace transform of the mismatched error may be approximated by

$$\bar{e}_{\tau,\tau_o}(s) \cong \bar{e}_{\tau,\tau}(s) + (\tau_o - \tau)p^2(s)e^{-2s\tau} \tag{2.13}$$

Adopting the notation

$$\mathcal{L}\{p_2(t)\} = p^2(s) \tag{2.14}$$

and taking inverse Laplace transforms throughout (2.13)

$$e_{\tau,\tau_o}(t) \cong e_{\tau,\tau}(t) + (\tau_o-\tau)p_2(t-2\tau) \tag{2.15}$$

It is noted that the error for the matched Smith scheme is a delayed version of the delay-free error (for the control scheme of Figure 1.1). Therefore, adopting the notation

$$e_F(t) = \mathcal{L}^{-1}\{\frac{1-p(s)}{s}\} \tag{2.16}$$

(2.15) can be written

$$e_{\tau,\tau_o}(t) \cong e_F(t-\tau) + (\tau_o-\tau)p_2(t-2\tau) \tag{2.17}$$

This is the approximation adopted by Marshall and Salehi (1982). The estimation technique relies on the simulation of delay-free curves $e_F(t)$ and $p_2(t)$, a process which is inexpensive in terms of computer time. By observation the estimate of the optimal model delay is then chosen, if possible, to be the value of τ_o which facilitates the cancellation of $e_F(t-\tau)$ with $(\tau_o-\tau)p_2(t-2\tau)$. The estimation technique is illustrated in the examples of the following section. The first part of each example is a delay-free optimisation involving the selection of a series controller. This determines the cost of the matched Smith scheme.

2.2: Examples of Performance Improvement by Mismatch

2.2.1: Example 2.1, A First-Order Example

In this example significant improvement is achieved by under-estimation of the plant delay. Consider the delay-free control scheme of Figure 1.1 incorporating a pure integrator

$$g(s) = \frac{1}{s} \tag{2.18a}$$

and a constant series controller

$$c(s) = k, \quad 0 < k \le 1 \tag{2.18b}$$

As in Section 1.2, the upper bound is imposed on the controller to ensure the example makes physical sense. The closed loop transfer function is given by

$$p(s) = \frac{k}{s+k} \quad , \quad 0 < k \le 1 \tag{2.18c}$$

and cost functional (1.1) by

$$J = \frac{1}{2k} \tag{2.18d}$$

Clearly, the delay-free optimisation is achieved when k takes its maximum value, $k = 1$, in which case $J = 0.5$.

Subplant (2.18a) is now considered to be part of a plant with delay τ, in either control or measurement. This plant is incorporated into the Smith scheme together with the series controller $c(s) = k = 1$. An estimate of the optimal model delay is to be found from the delay-free simulation of $e_F(t)$ and $p_2(t)$, shown for this example in Figure 2.1a.

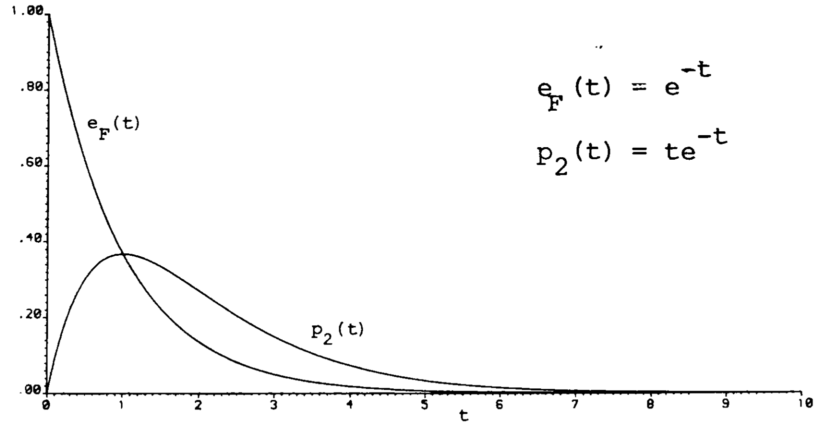


FIGURE 2.1a: The Delay-Free Curves; First-Order Example

If $(\tau_o-\tau)p_2(t-2\tau)$ is to cancel $e_F(t-\tau)$ when they are added, it is clear that $(\tau_o-\tau)p_2(t-2\tau)$ must be negative. In other words, $\tau_o-\tau$ must be negative, or equivalently, the model delay must be an underestimate of the plant delay. Consider the plant delay $\tau = 1.0$. The curves of Figure 2.1a shifted by the amounts indicated in expression (2.17) are shown in Figure 2.1b.

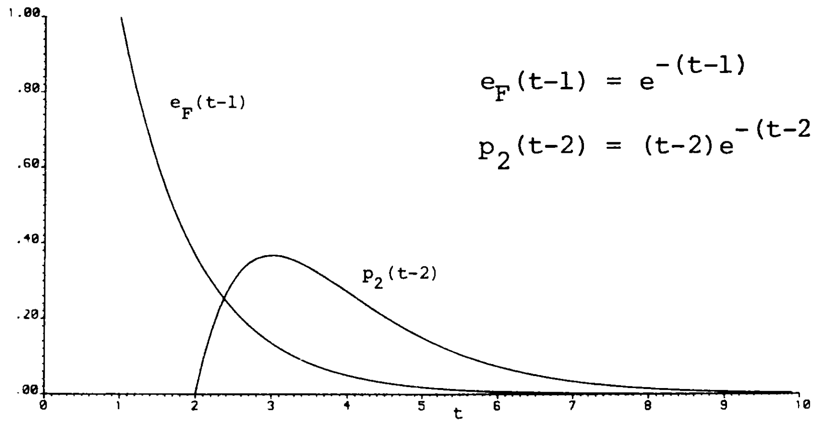


FIGURE 2.1b: Shifted Version of the Delay-Free Curves; First-Order Example

The curve $p_2(t-2)$ reaches a peak of e^{-1} when $t=3$. Observing the nature of $p_2(t-2)$ and $e_F(t-1)$ the best cancellation would appear to take place when the two curves are scaled to cancel exactly at $t=3$.

In other words, it is required to choose τ_o to satisfy

$$e^{-2} + (\tau_o - 1)e^{-1} = 0 \tag{2.18e}$$

with the result that $\tau_o = 0.63$, an underestimate of the plant delay. The accuracy of this estimate can now be checked against Figure 2.2.

The family of curves in Figure 2.2 shows the variation in cost with mismatch for this first-order example.

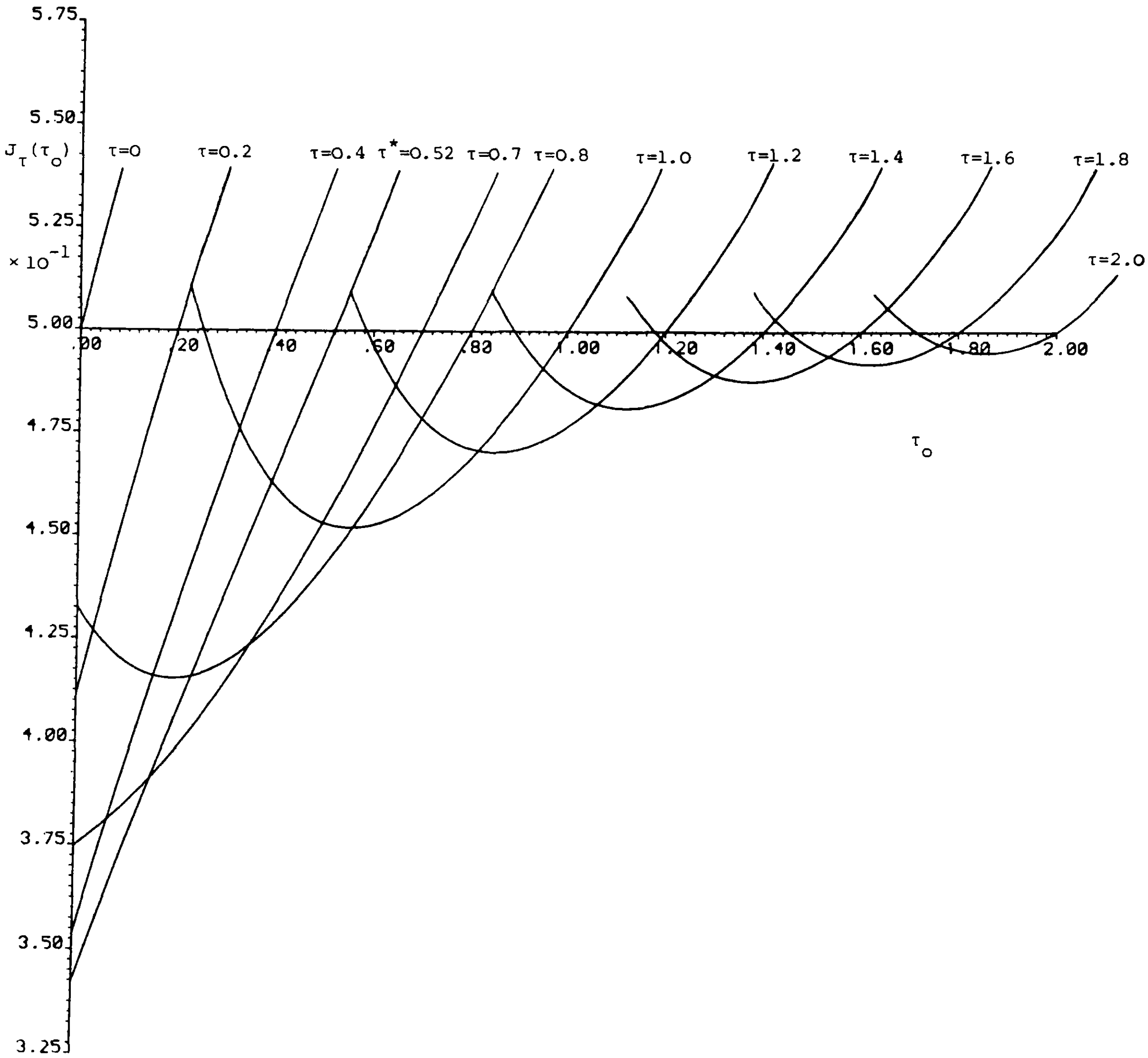


FIGURE 2.2: Variation in Cost with Mismatch; First-Order Example

Each curve corresponds to a particular value of plant delay. As anticipated by the estimation technique, for each plant delay improvement is by underestimation. For plant delay $\tau = 1.0$ the estimation technique is seen to work well, the optimal model delay being $(\tau_o)_{opt} = 0.56$. The percentage improvement $I(\tau,\tau_o)$ is defined by the formula

$$I(\tau,\tau_o) = \left\{\frac{J - J_\tau(\tau_o)}{J}\right\} \times 100\% \tag{2.19}$$

For the case of $\tau = 1$, the percentage improvement with the optimal model delay is $I(1,0.56) = 15.5\%$. The maximum percentage improvement is achieved with a plant delay $\tau^* = 0.52$ for which the optimal model delay $(\tau^*_o)_{opt} = 0$ yields an improvement of 31.5%.

The mismatched Smith scheme with zero model delay is an interesting special case. The transfer function of the delay mismatched Smith scheme may be expressed as

$$\frac{\bar{y}_{\tau,\tau_o}(s)}{\bar{r}(s)} = \frac{c(s)g(s)e^{-s\tau}}{1 + c(s)g(s) + c(s)g(s)(e^{-s\tau} - e^{-s\tau_o})} \tag{2.20}$$

When the model delay is zero this reduces to

$$\frac{\bar{y}_{\tau,o}(s)}{\bar{r}(s)} = \frac{c(s)g(s)e^{-s\tau}}{1 + c(s)g(s)e^{-s\tau}} \tag{2.21}$$

the transfer function of a control scheme comprising unity negative feedback around the series controller and the plant. In Example 2.1 extensive numerical calculation shows that for $\tau \leq 0.74$, $(\tau_o)_{opt} = 0$. In other words, for plant delays small enough the mismatched Smith scheme does not improve on the unity negative feedback scheme. It is noted in passing, that mismatch in the subplant does not prevent the Smith scheme with zero model delay reducing to the unity negative feedback scheme. The transfer function of the Smith scheme with mismatch in delay and subplant is given by

$$\frac{\bar{y}_o(s)}{\bar{r}(s)} = \frac{c(s)g(s)e^{-s\tau}}{1 + c(s)g_o(s) + c(s)(g(s)e^{-s\tau} - g_o(s)e^{-s\tau_o})} \tag{2.22}$$

which reduces to (2.21) when the model delay is zero. The next example to be considered is a more extensive examination of the example of Chapter 1, where improvement has been achieved by overestimation of plant delay. Circumstances are discovered where the example is marginally improved by underestimation of plant delay.

2.2.2: Example 2.2, A Second-Order Example

Subplant (1.12a) is considered to be part of a plant with delay τ in either control or measurement. This plant is incorporated into the Smith scheme together with the series controller $c(s) = k = 1$. For the plant delays $\tau = 1$ and $\tau = 5$ the estimation technique is implemented to find approximate values for the corresponding optimal model delays. The curves $e_F(t)$ and $p_2(t)$ for this example are given in Figure 2.3a.

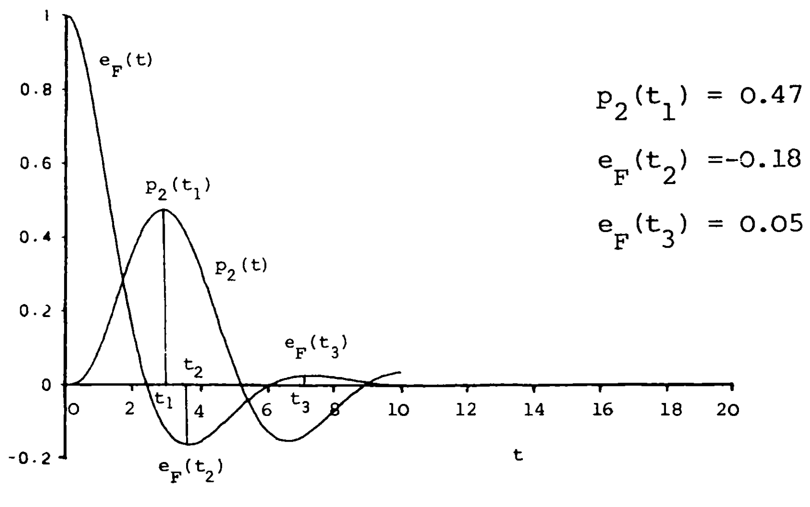


FIGURE 2.3a: The Delay-Free Curves; Second-Order Example

The times $t_1 = 3$ secs and $t_3 = 7.2$ secs denote the occurrence of the largest peaks of $p_2(t)$ and $e_F(t)$ respectively. Time $t_2 = 3.6$ secs denotes the occurrence of the largest trough of $e_F(t)$.

When $\tau = 1$ the curves of Figure 2.3a shifted by the relevant delays are shown in Figure 2.3b.

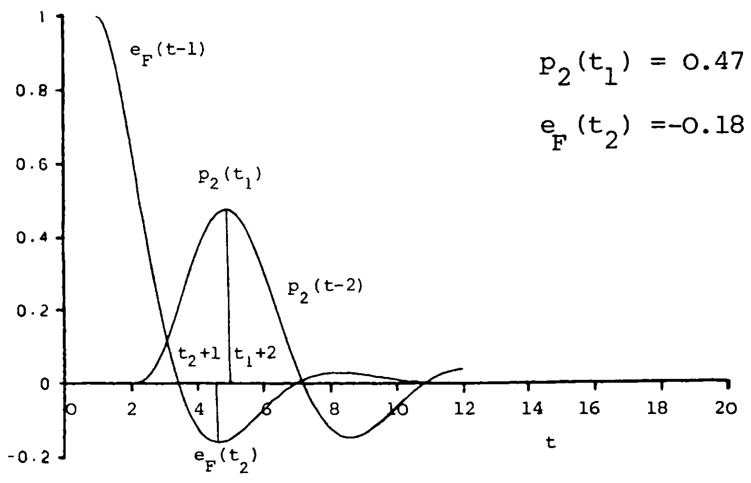


FIGURE 2.3b: Shifted Version of Delay-Free Curves; Second-Order Example, $\tau = 1$.

The peaks of $p_2(t-2)$ occur at approximately the same times as the troughs of $e_F(t-1)$. Therefore, the estimate of the optimal model delay is chosen to satisfy

$$\tau_o - \tau = \frac{-e_F(t_2)}{p_2(t_1)} \tag{2.23a}$$

with the result that $\tau_o = 1.38$, an overestimate of the plant delay. The accuracy of the estimation technique is seen by referring to Section 1.2 where the optimal model delay is given as $(\tau_o)_{opt} = 1.32$.

It is observed that for large enough time delays, the shifted versions of the delay-free curves will have a good correspondence between their peaks. In this instance, the example may be improved by underestimation of plant delay. For the plant delay $\tau = 5$ the curves $e_F(t-5)$ and $p_2(t-10)$ take the form shown in Figure 2.3c.

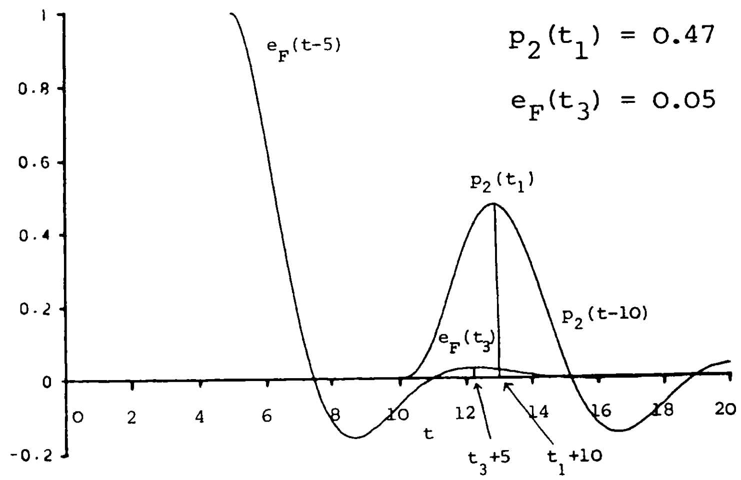


FIGURE 2.3c: Shifted Versions of the Delay-Free Curves; Second-Order Example, $\tau = 5$

The first peak of $e_F(t-5)$ occurs at approximately the same time as the first peak of $p_2(t-10)$. The choice of $\tau_o - \tau$ for the required cancellation is therefore

$$\tau_o - \tau = \frac{-e_F(t_3)}{p_2(t_1)} \tag{2.23b}$$

which results in $\tau_o = 4.9$, an underestimate of the plant delay.

The family of curves in Figure 2.4 shows the variation in cost with mismatch for this second-order example.

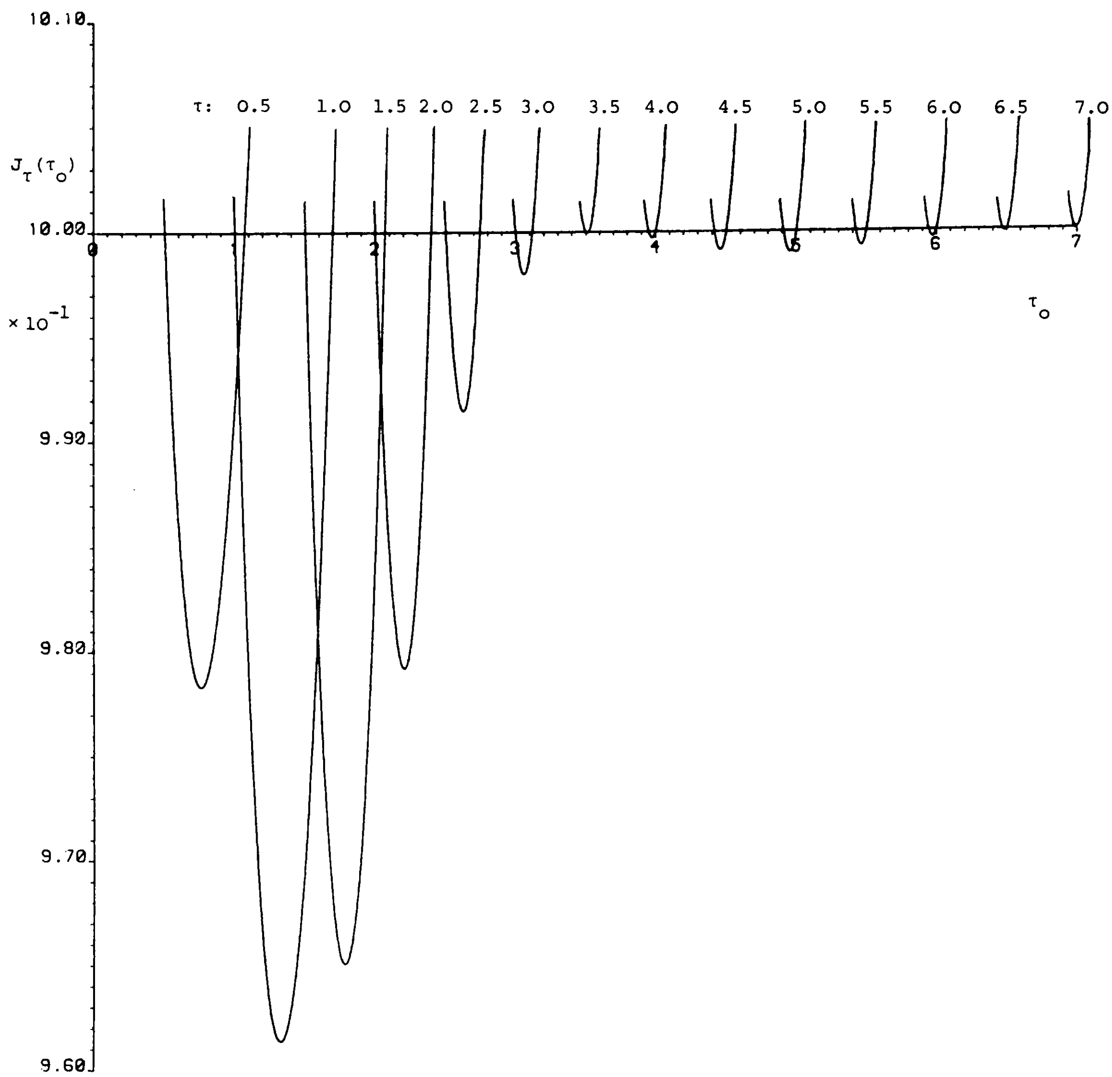


FIGURE 2.4: Variation in Cost with Mismatch; Second-Order Example

Figure 2.4 shows that for each plant delay in the approximate range $0 < \tau < 3.5$ improvement is by overestimation of plant delay. Figure 2.3b indicates that underestimation will impair performance for this range of plant delay as it causes the first two terms of the mismatched error to add rather than cancel.

For the particular case of $\tau = 5$, Figure 2.4 shows that improvement is indeed by underestimation, the optimal model delay, $(\tau_o)_{opt} = 4.94$, producing an improvement of 0.1%. Furthermore, for plant delays in the approximate range $3.5 < \tau < 7.0$ any improvement will be by underestimation. Figure 2.3c indicates that overestimation will impair performance for this range of plant delay, as it causes terms of the mismatched error to add rather than cancel. It also suggests that the reason why only a small improvement in performance is achieved is that only a small peak of the error is being cancelled. In practical terms, this improvement is too small to be significant. Furthermore, as the plant delay becomes large, to obtain any improvement over the matched case, a very accurate choice of model delay is required. However, this example shows in principle that improvement may be by overestimation or underestimation depending on the size of the plant delay.

If an oscillatory error could be found which decays more slowly to zero, with consecutive peaks and troughs of similar magnitude, it may be possible to construct an example where depending on plant delay, significant improvement is possible by both underestimation and overestimation. Such an error is provided by the following third-order example.

2.2.3: Example 2.3; A Third-Order Example

The delay-free control scheme of Figure 1.1 is taken to incorporate the subplant

$$g(s) = \frac{1}{s(s^2+2s + 16)} \tag{2.24a}$$

and a constant series controller

$$c(s) = k\ , \quad 0 < k \le 20 \tag{2.24b}$$

The upper bound is again imposed to ensure the example makes sense from a physical point of view. The characteristic equation of the control scheme

$$s^3 + 2s^2 + 16s + k = 0 \tag{2.24c}$$

has all three roots in the left-half plane. As the control scheme is stable it can be shown analytically that cost functional (1.1) is given by

$$J = \frac{8}{k} + \frac{2}{32-k} \tag{2.24d}$$

For the given range of k, the delay-free optimisation is achieved when k takes its maximum value, $k = 20$, in which case $J = 0.5666$.

Subplant (2.24a) is now considered to be part of a plant with delay τ in either control or measurement. This plant is incorporated into the Smith scheme together with the series controller $c(s) = k = 20$. The curves $e_F(t)$ and $p_2(t)$ for this example are shown in Figure 2.5a. The times $t_1 = 1.26$ seconds and $t_2 = 1.86$ seconds denote the occurrence of the first (and largest) peaks of $p_2(t)$ and $e_F(t)$ respectively. The time $t_3 = 2.66$ seconds is the occurrence of the second and largest trough of the error.

When the plant delay is chosen to be the difference between times t_1 and t_2 it takes the value $\tau = 0.6$. The curves of Figure 2.5a shifted by the relevant amounts are shown in Figure 2.5b.

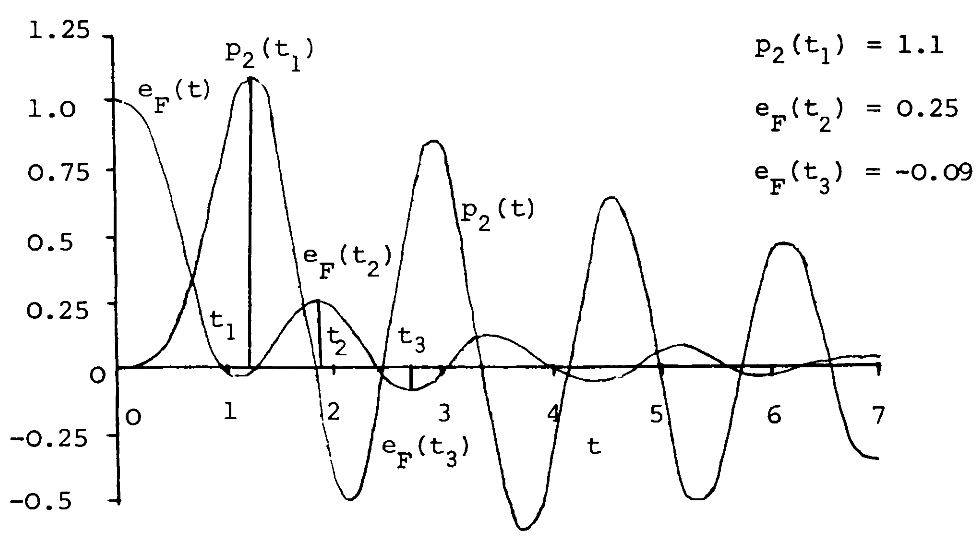


FIGURE 2.5a: The Delay-Free Curves; Third-Order Example

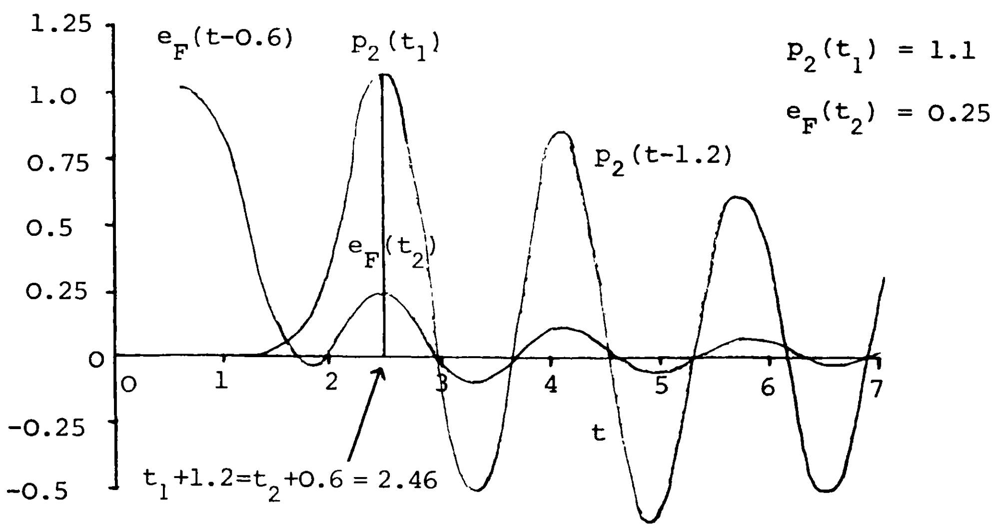


FIGURE 2.5b: Shifted Version of the Delay-Free Curves; Third-Order Example, $\tau = 0.6$

The choice of plant delay ensures the first peak of $e_F(t-0.6)$ occurs at exactly the same time as the first peak of $p_2(t-1.2)$. As a bonus, the times of subsequent peaks are approximately equal. This correspondence simplifies the choice of $\tau_o - \tau$

$$\tau_o - \tau = \frac{-e_F(t_2)}{p_2(t_1)} \tag{2.24e}$$

which results in $\tau_o = 0.37$, an underestimate of the plant delay.

The estimation technique is now used a second time to produce circumstances where improvement may be by overestimation. When $\tau = t_3 - t_1 = 1.4$ the curves $e_F(t-1.4)$ and $p_2(t-2.8)$ take the form of Figure 2.5c

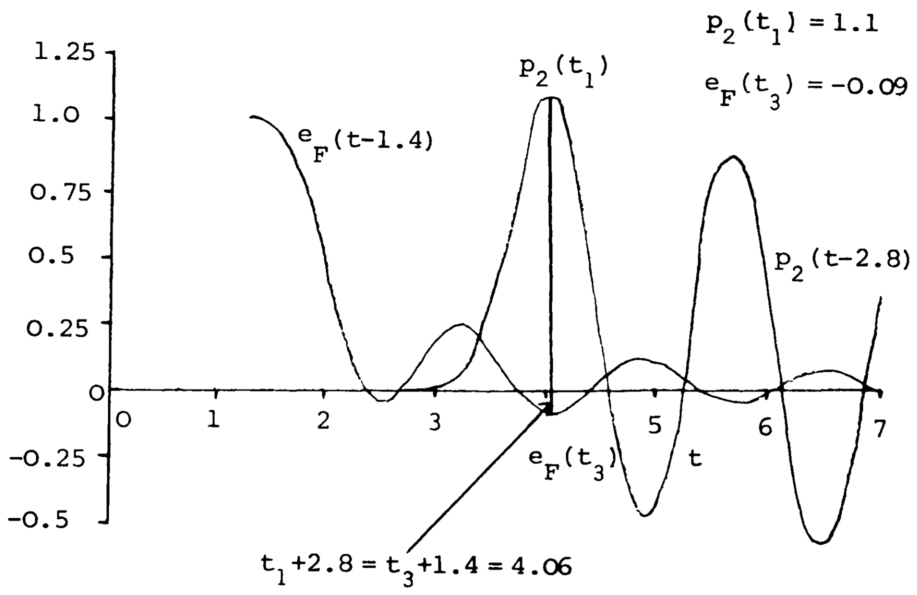


FIGURE 2.5c: Shifted Version of the Delay-Free Curves; Third-Order Example, $\tau = 1.4$

The plant delay ensures the largest trough of $e_F(t-1.4)$ occurs at exactly the same time as the first peak of $p_2(t-2.8)$. As a bonus, subsequent peaks of $p_2(t-2.8)$ occur at approximately the same times as troughs of $e_F(t-1.4)$. Therefore, the estimate of the optimal model delay is chosen to satisfy

$$\tau_o - \tau = \frac{-e_F(t_3)}{p_2(t_1)} \tag{2.24f}$$

which results in $\tau_o = 1.48$, an overestimate of the plant delay.

The family of curves in Figure 2.6 shows the variation in cost with mismatch for this third-order example.

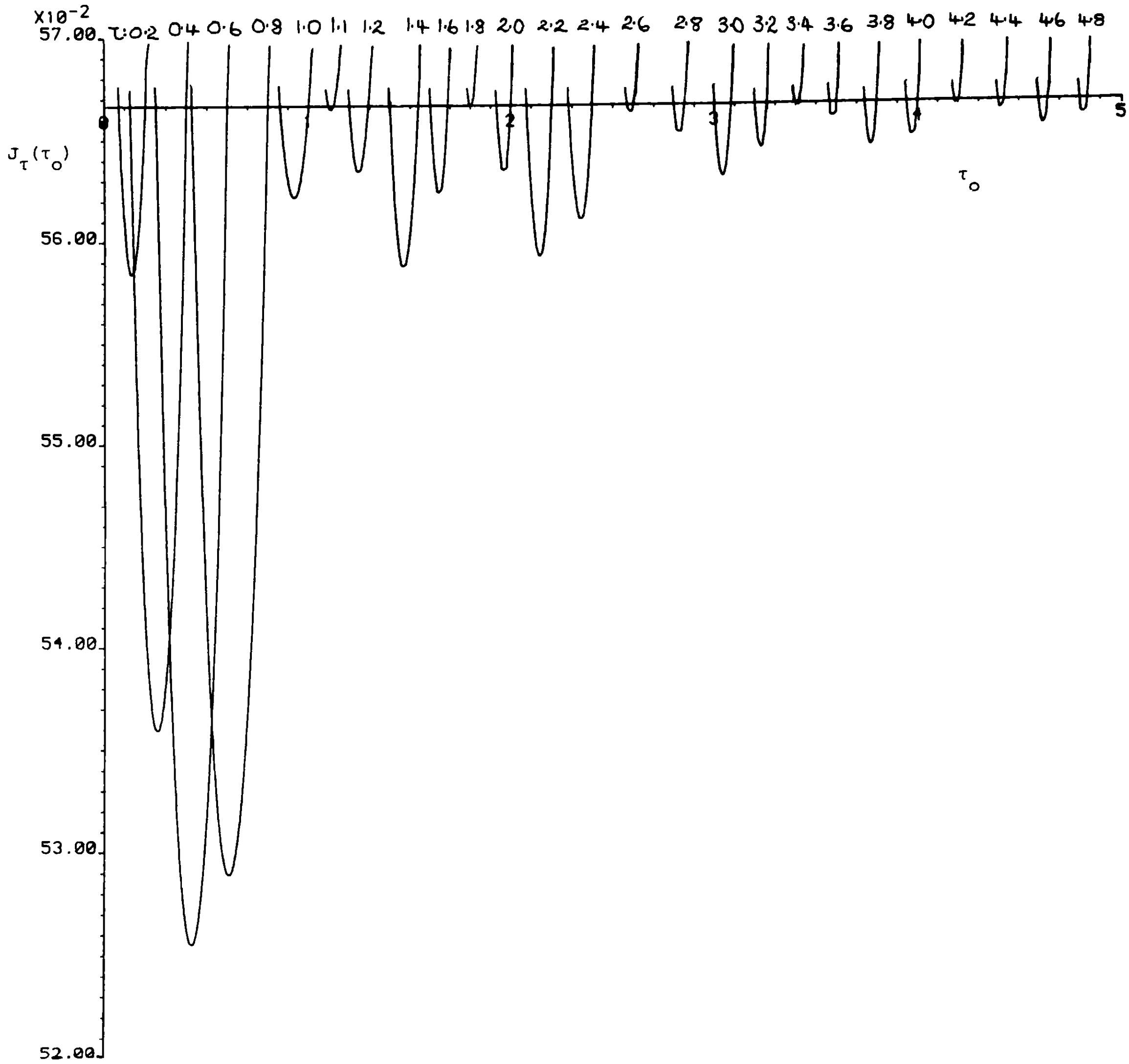


FIGURE 2.6: Variation in Cost with Mismatch; Third-Order Example

The numerical details are summarised in Table 2.1 below:

Approximate Range of Plant Delay	Plant Delay Yielding Maximum Improvement	Corresponding Optimal Model Delay	Performance Improvement
(0, 1.1]	0.6	0.43	7.25%
[1.1, 1.8]	1.4	1.46	1.38%
[1.8, 2.6]	2.2	2.14	1.30%
[2.6, 3.4]	3.0	3.04	0.61%
[3.4, 4.2]	3.8	3.77	0.35%
[4.2, 5.0]	4.6	4.62	0.19%

TABLE 2.1: Numerical Details for the Third-Order Example

For the values of delay considered in detail the estimation technique again works well. When $\tau = 0.6$, $(\tau_o)_{opt} = 0.43$ producing a percentage improvement by underestimation of over 7%. For $\tau = 1.4$, $(\tau_o)_{opt} = 1.46$ producing a percentage improvement by overestimation of approximately 1.4% The greater improvement for $\tau = 0.6$ rather than $\tau = 1.4$ is consistent with the first peak of $e_F(t)$ being greater than its second trough. It is observed that as the plant delay is increased from zero improvement is achieved alternately by underestimation and overestimation. This is consistent with the change in plant delay allowing the peaks of $p_2(t-2\tau)$ to alternately coincide with the peaks and troughs of $e_F(t-\tau)$. For plant delays greater than $\tau = 5$ no noticeable improvement in performance is achieved by mismatch.

The results of the examples considered can be summarised as follows. Improvement is by underestimation or overestimation depending on whether mismatch is being utilised to cancel a peak or trough of the matched error.

Furthermore, the size of the resulting improvement is determined by the size of the peak or trough being cancelled. This gives insight into why most results on this topic produce improvement by overestimation. The earlier papers contain second-order subplants, for which the error resulting from the step response is of the form given in Figure 2.3a. In particular, the first trough is greater than any subsequent peak or trough. Moreover, the plant delay is such that it is this first trough which is cancelled, thus producing improvement by overestimation.

2.3: The Augmented Smith Scheme

The fact that the potential for improvement by mismatch depends on the plant delay is illustrated in each of the three examples of the previous section. Potential improvement is defined as a percentage of the matched cost by the formula

$$T(\tau) = I(\tau, (\tau_o)_{opt}) \qquad (2.25)$$

As potential improvement depends on the plant delay, adding time delays into the Smith scheme may allow further improvement in performance by mismatch. The symbol τ^* denotes the plant delay for which the potential improvement by mismatch is greatest. For a Smith scheme with plant delay τ, where $\tau < \tau^*$, it is hoped to include a delay $\tau^* - \tau$, so the cost of the augmented Smith scheme varies with model delay in the same way as the Smith scheme incorporating plant delay τ^*.

The cost functional for the mismatched Smith scheme for plant delay τ^*, model delay τ_o^* and input $r(t)$ is

$$J_{\tau^*}(\tau_o^*) = \int_{\tau^*}^{\infty} \{r(t-\tau^*) - y_{\tau^*,\tau_o^*}(t)\}^2 h(t-\tau^*)dt \tag{2.26}$$

By a change of variable (2.26) can be expressed as the cost functional for a control scheme with plant delay τ

$$J_{\tau^*}(\tau_o^*) = \int_{\tau}^{\infty} \{r(t-\tau) - y_{\tau^*,\tau_o^*}(t-(\tau-\tau^*))\}^2 h(t-\tau)dt \tag{2.27}$$

In other words, the cost of the mismatched Smith scheme for plant delay τ^*, input $r(t)$ and output $y_{\tau^*,\tau_o^*}(t)$, is equal to the cost of a control scheme for plant delay τ which produces output $y_{\tau^*,\tau_o^*}(t-(\tau-\tau^*))$ from input $r(t)$. By (2.1) this output has Laplace transform

$$\mathcal{L}\{y_{\tau^*,\tau_o^*}(t-(\tau-\tau^*))\} = \frac{p(s)e^{-s\tau}\bar{r}(s)}{1+p(s)(e^{-s\tau^*}-e^{-s\tau_o^*})} \tag{2.28}$$

and is produced by the control scheme of Figure 2.7 when $\tau_o = \tau_o^*$.

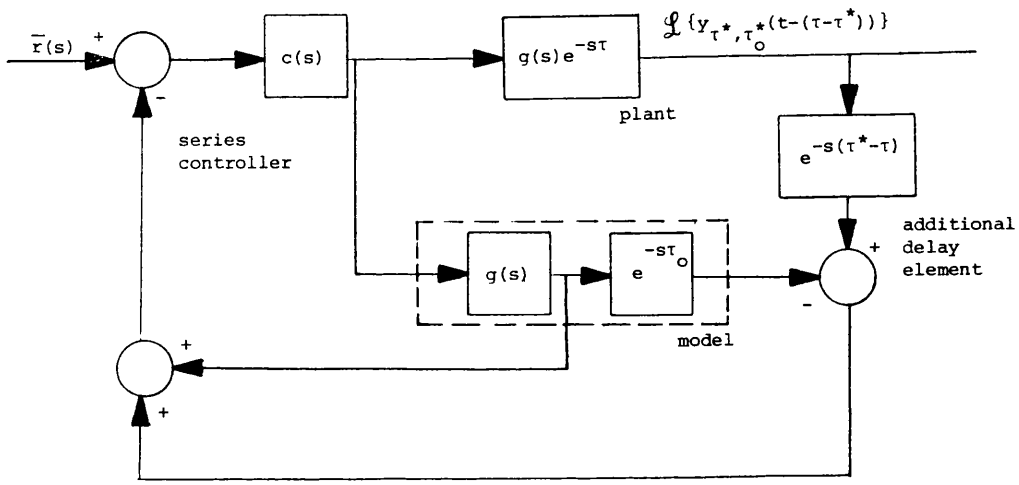


FIGURE 2.7: The Augmented Smith Control Scheme

Therefore, for the same choice of model delay, the cost of the augmented Smith scheme for plant delay τ is the same as that of the mismatched Smith scheme for plant delay τ^*. Hence, to achieve the maximum potential improvement by mismatch for the Smith scheme with plant delay $\tau, \tau < \tau^*$, add the delay $\tau^* - \tau$ as in Figure 2.7 and choose the model delay to be $\tau_o = (\tau_o^*)_{opt}$.

Example 2.3 of Section 2.2.3 provides a numerical illustration of the advantage of the augmented Smith scheme over the ordinary Smith scheme. Consider the plant delay $\tau = 0.2$ seconds. Figure 2.6 shows that for the mismatched Smith scheme the optimal model delay is $(\tau_o)_{opt} = 0.13$ which produces an improvement over the matched cost of approximately 1.5%. However, incorporating an additional delay $\tau^* - \tau = 0.4$ as in Figure 2.7, the augmented Smith scheme with $\tau_o = (\tau_o^*)_{opt} = 0.43$ produces an improvement of over 7%.

It appears the technique is not applicable for plants with delays greater than τ^* as the delay term $e^{-s(\tau^* - \tau)}$ is no longer realisable. However, some improvement may be possible. Consider Example 2.3 of Section 2.2.3. Figure 2.6 shows that for the range of plant delays [1.1,1.8] any improvement is by overestimation, with the greatest potential improvement occurring at $\tau^\dagger = 1.4$. Furthermore, for the plant delay $\tau = 1.1$ no improvement is possible via the mismatched Smith scheme. Whilst it is impossible to subtract delays to obtain the behaviour of $\tau^* = 0.6$, adding $\tau^\dagger - \tau = 0.3$ in place of $\tau^* - \tau$ produces the behaviour of $\tau^\dagger = 1.4$. In this way, the augmented Smith scheme provides an improvement of 1.38%

An alternative and perhaps more intuitive location for the additional time delay is the outer feedback loop (Marshall and Salehi, 1982), see Figure 2.8.

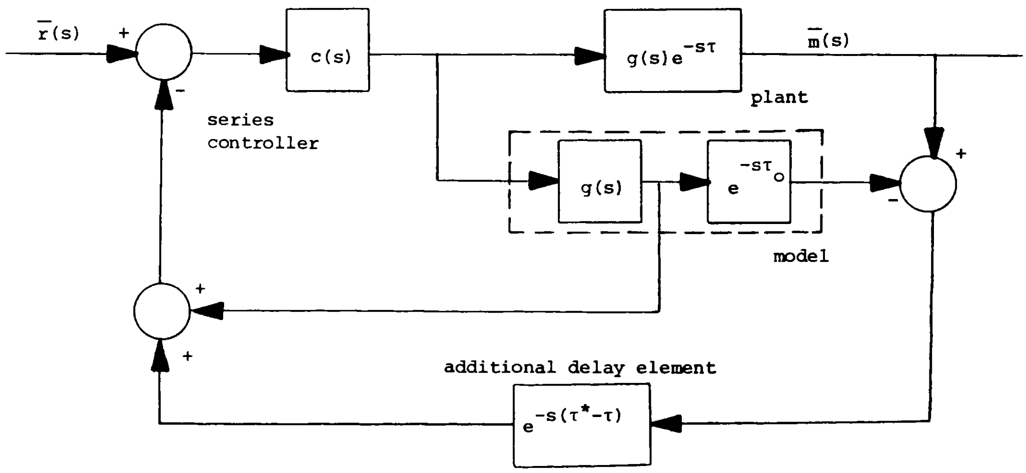


FIGURE 2.8: An Alternative Augmented Smith Scheme

However, it is seen from the following analysis, that the model delay required by the scheme of Figure 2.8 may be negative and hence unrealisable. The augmented Smith scheme of Figure 2.8 for plant delay τ, $\tau < \tau^*$, and model delay τ_o, produces output m(t) from input r(t) where

$$\bar{m}(s) = \frac{p(s)e^{-s\tau}\bar{r}(s)}{1 + p(s)(e^{-s\tau^*} - e^{-s(\tau_o+\tau^*-\tau)})} \tag{2.29}$$

The Laplace transform of the output for the mismatched Smith scheme with plant delay τ^*, model delay τ_o^* and input r(t) is

$$\bar{y}_{\tau^*,\tau_o^*}(s) = \frac{p(s)e^{-s\tau^*}\bar{r}(s)}{1 + p(s)(e^{-s\tau^*} - e^{-s\tau_o^*})} \tag{2.30}$$

From the analysis of (2.26), (2.27) and (2.28), the costs of the control schemes of Figures 2.7 and 2.8 are equal when the denominators of (2.29) and (2.30) are equal. In other words, the relevant costs are equal when

$$\tau_o = \tau_o^* - \tau^* + \tau \tag{2.31}$$

Therefore, to obtain the maximum potential improvement by mismatch the model delay required by the scheme of Figure 2.8 is

$$\tau_o = ((\tau_o^*)_{opt} - \tau^*) + \tau \tag{2.32}$$

If the Smith scheme incorporating plant delay τ^* is optimised by under-estimation to the extent that $\tau^* - (\tau^*_o)_{opt} > \tau$ then the model delay determined by (2.32) is negative and hence unrealisable. For the third-order example of Section 2.2.3, $\tau^* = 0.6$ and $(\tau^*_o)_{opt} = 0.43$. Therefore, for any plant delay satisfying $\tau < 0.17$ the model delay required for Figure 2.8 is unrealisable. For this reason the augmented Smith scheme presented in Figure 2.7 is preferable to the scheme of Figure 2.8. Now that the performance of a time-delay control scheme has been improved by the addition of a time delay, it is of interest to discover if similar techniques will provide improvement in delay-free schemes.

2.4: The Improvement of Delay-Free Control Schemes by the Addition of Time-Delays

Figure 2.9 shows the delay-free control scheme of Figure 1.1 with an additional feedback loop containing two time delays. It is observed that Figure 2.9 reduces to Figure 1.1 when $\tau = \tau_o$.

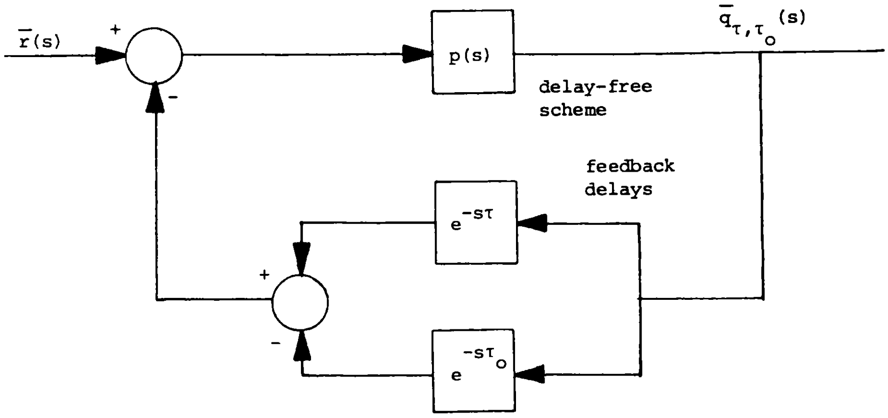


FIGURE 2.9: Delay-Free Scheme with Additional Delayed Feedback Loop

The transfer function of the control scheme of Figure 2.9 has a familiar form

$$\frac{\bar{q}_{\tau,\tau_o}(s)}{\bar{r}(s)} = \frac{p(s)}{1 + p(s)(e^{-s\tau} - e^{-s\tau_o})} \tag{2.33}$$

the output being an advanced version of the mismatched Smith output for plant delay τ and model delay τ_o

$$q_{\tau,\tau_o}(t) = y_{\tau,\tau_o}(t+\tau) \tag{2.34}$$

Therefore, the cost C of the control scheme of Figure 2.9 is equal to that of the mismatched Smith scheme for plant delay τ and model delay τ_o

$$C = \int_o^\infty (r(t) - q_{\tau,\tau_o}(t))^2 \, dt$$

$$= \int_o^\infty (r(t) - y_{\tau,\tau_o}(t+\tau))^2 dt$$

$$= \int_\tau^\infty (r(t-\tau) - y_{\tau,\tau_o}(t))^2 h(t-\tau) dt$$

$$= J_\tau(\tau_o) \tag{2.35}$$

As the cost of the mismatched Smith scheme is minimised by parameter values $\tau = \tau^*$ and $\tau_o = (\tau^*_o)_{opt}$, rather than the matched values, so is the cost of the scheme of Figure 2.9. Clearly, the introduction of delay into the delay-free scheme has produced the desired reduction in cost.

Suh and Bien (1979, 1980) attempt to improve delay-free schemes by replacing derivative elements with time delays. Using an ITAE cost functional Suh and Bien (1980) compare their delay compensator scheme of Figure 2.10, with a proportional-plus-derivative compensator scheme.

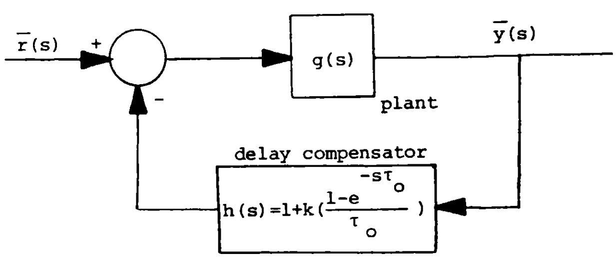


FIGURE 2.10: The Control Scheme of Suh and Bien

For an example with a second-order subplant, numerical optimisation shows the delay compensator scheme to be superior.

A comparison of the respective transfer functions shows the relation between the scheme of Suh and Bien and the Smith scheme with extra gain of Figure 2.11.

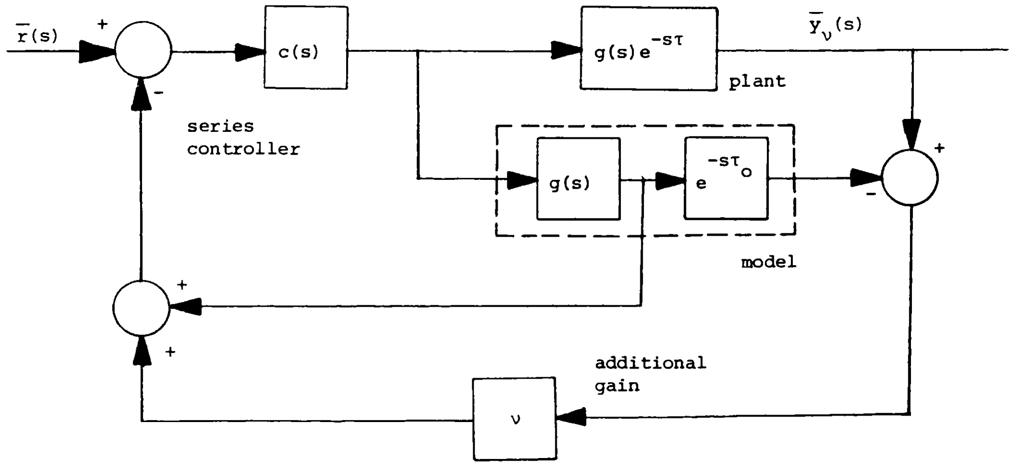


FIGURE 2.11: The Mismatched Smith Scheme with an Extra Gain

The transfer function of the scheme of Suh and Bien is

$$\frac{\bar{y}(s)}{\bar{r}(s)} = \frac{g(s)}{1 + g(s)\left(1+k\left(\frac{1-e^{-s\tau_o}}{\tau_o}\right)\right)} \tag{2.36}$$

whereas the transfer function of the scheme of Figure 2.11 is

$$\frac{\bar{y}_\nu(s)}{\bar{r}(s)} = \frac{c(s)g(s)e^{-s\tau}}{1+c(s)g(s) + \nu c(s)(g(s)e^{-s\tau}-g(s)e^{-s\tau_o})} \tag{2.37}$$

In the special case where $\tau = 0$, $c(s) = 1$ and $\nu = k/\tau_o$, (2.37) reduces to (2.36).

The control scheme of Suh and Bien is therefore the mismatched Smith scheme with zero plant delay, $c(s) = 1$, and a gain k/τ_o in the outer feedback loop.

After a considerable numerical calculation Suh and Bien (1980) determine parameters τ_o^* and k^* for which the control scheme of Figure 2.10 produces the greatest improvement over the proportional-plus-derivative scheme. The identical improvement can be achieved by implementing the Smith scheme with extra gain, for the delay-free subplant, when $c(s) = 1$, $\tau_o = \tau_o^*$ and $\nu = k^*/\tau_o^*$.

Conclusions

The error of the mismatched Smith scheme may be expressed as an infinite series, the first term of which is the matched error. Improvement by mismatch occurs when an appropriate choice of the model delay allows this first term to be cancelled by subsequent terms. This produces a mismatched error "close" to the zero function, which ensures the mismatched Smith scheme a reduced cost. A technique to estimate the optimal model delay leads to the conjecture that improvement is by underestimation or overestimation depending on whether mismatch is being utilised to cancel a peak or a trough of the matched error. Furthermore, the size of the resulting improvement is determined by the size of the peak or trough being cancelled.

General engineering practice is to overestimate unknown time delays. This view is supported by the literature, where in the main, second-order subplants are considered. The significant feature of the error of a second-order subplant is that its first trough is greater than any subsequent peak or trough. Therefore, improvement is most readily obtained when the plant delay is such that this first trough is cancelled. This gives insight into why most results in this area favour the practice of overestimating the plant delay. This chapter contains examples which may be optimised by underestimation. For the first-order example it is always better to underestimate, whereas for the second- and third-order examples improvement is possible by both underestimation and overestimation depending on the size of the plant delay. It is also noted that for a sufficiently large plant delay, very accurate modelling of the delay is required.

If the criterion for judging performance satisfies the engineering requirements of the problem, the addition of the relevant time delay into the

Smith scheme to artificially increase the plant delay may allow further improvement by mismatch. The choice of model delay required to achieve the maximum potential improvement by mismatch is determined for two different locations of this additional time delay. The advantage of the less intuitive location is that the required model delay is always realisable. Finally, the performance of delay-free schemes is improved by the addition of a feedback loop containing the difference of two time delays. The optimal values of the two delays are determined from the earlier analysis of associated time delay schemes.

Chapter 3: The Optimal Control of Linear Systems with Control Time Delays

Chapter 3: The Optimal Control of Linear Systems with Control Time Delays

Introduction

The ISE cost functional examined in Chapters 1 and 2 is optimised parametrically. The extra parameters made available by the presence of mismatch introduce the possibility of the cost of the mismatched Smith scheme being less than the corresponding matched cost. In the remaining chapters of this thesis, a quadratic cost functional is associated with the time delay plant. As the cost functional is quadratic, a matched predictor control scheme is known to be optimal.

The first section of the chapter, Section 3.1, is a discussion of delay-free Linear Quadratic Performance (LQP) problems, the results of which extend easily to plants with time delays. Following the work of Fuller (1968) and Mee (1973), Marshall, Ireland and Garland (1977) present an optimal control scheme for plants with delay in control, an analysis of which is given in Section 3.2. Section 3.3 develops its time-varying extension. The structure of the control schemes is similar to that of the Smith scheme, containing a plant model for purposes of prediction. The analysis throughout Chapter 3, and the subsequent chapters, is in the time domain using the state-space formulation.

3.1: Linear Quadratic Performance Control Problems

A comprehensive treatment of LQP problems can be found in many texts (Athans and Falb, 1960; Lee and Markus, 1967; Brockett, 1970; Kwakernaak and Sivan, 1972; Wonham, 1979; Jacobson et al, 1980) consequently, this section is restricted to statements of two problems together with their solutions. The first problem involves a time-varying linear system with a time-varying cost functional which is optimised over a fixed, finite time interval. The solution is a time-varying, linear feedback of the current state. In the second problem the linear system and cost functional are time-invariant and the optimisation is over the semi-infinite time horizon. In this instance, the solution is also of linear feedback form but has the advantage of being time-invariant.

For the first problem, a noise-free, time-varying, linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad , \quad x(t_o) = x_o \tag{3.1}$$

is defined on a fixed, finite time interval $[t_o,t_f]$, where $A(t)$ is an $n \times n$ matrix, $B(t)$ is an $n \times m$ matrix, $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$, and it is required to find the control which minimises the quadratic cost functional

$$J(u) = \langle x(t_f), Gx(t_f)\rangle + \int_{t_o}^{t_f} \{\langle x(\sigma), Q(\sigma)x(\sigma)\rangle + \langle u(\sigma), R(\sigma)u(\sigma)\rangle\} d\sigma \tag{3.2}$$

where G and Q are symmetric, positive semi-definite matrices $(G,Q(t) \geq 0)$, $R(t)$ is a symmetric positive definite matrix $(R(t) > 0)$ and t_f is the fixed final time.†

The first term of the cost functional penalises any deviations from the desired final state, and during the control action, the first and second terms of the integrand penalise the state of the system, and the

† Throughout the thesis subscript zero is used to denote mismatched quantities, except in the cases of x_o and t_o where it is standard notation for initial conditions.

use of control, respectively. However, the method of quadratic optimisation has been widely adopted for reasons other than any strict physical interpretation of the cost functional. These include the fact that the optimal control is easily calculated and implemented, and that by judicious choice of cost functional parameters, a good dynamic response can be achieved.

As the control is unconstrained, the solution to the above problem is obtained straightforwardly from the Bellman, Hamilton, Jacobi equation. The unique optimal control which minimises (3.2) subject to (3.1) is a time-varying, linear feedback of the current state

$$u(t) = -R^{-1}(t)B^T(t)K(t)x(t) \quad , \quad t_o \le t \le t_f \tag{3.3}$$

where K(t) is the unique solution of the matrix differential Riccati equation

$$\dot{K}(t) + K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) + Q(t) = 0 \qquad t_o \le t \le t_f \tag{3.4a}$$

together with the boundary condition

$$K(t_f) = G \tag{3.4b}$$

The minimum cost associated with the optimal control is

$$J(u(t)) = \langle x_o, K(t_o)x_o \rangle \tag{3.5}$$

For the second problem, attention is restricted to the time-invariant versions of the above equations. A noise-free, time-invariant, linear system is given by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_o \tag{3.6}$$

where A is a constant $n \times n$ matrix, B is a constant $n \times m$ matrix, $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$, and it is required to find the control which minimises the time-invariant quadratic cost functional

$$J(u) = \int_{0}^{\infty} \{\langle x(\sigma), Qx(\sigma)\rangle + \langle u(\sigma), Ru(\sigma)\rangle\} d\sigma \tag{3.7}$$

where $Q \geq 0$ and $R > 0$ are symmetric constant matrices. The solution to this second problem takes the following form; if there exists a control for which (3.7) is finite, then there exists an optimal control given by

$$u(t) = -R^{-1}B^{T}Kx(t) \quad , \quad t \geq 0 \tag{3.8}$$

where K is a symmetric and positive definite solution of the algebraic Riccati equation

$$KA + A^{T}K - KBR^{-1}B^{T}K + Q = 0 \tag{3.9}$$

The minimum cost associated with the optimal control is

$$J(u(t)) = \langle x_o, Kx_o\rangle \tag{3.10}$$

The above result requires the existence of a control which yields a finite value of cost (3.7). An easily satisfied condition for the existence of such a control is that the pair (A,B) is stabilizable. The results of Section 3.1 are now applied to time-delay plants in Sections 3.2 and 3.3.

3.2: An Optimal Control Scheme for Time-Invariant Subplants

Subplant (3.6) is now considered to be part of a plant with delay τ in control, see Figure 3.1

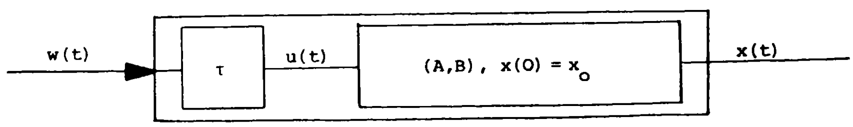


FIGURE 3.1: The Plant, incorporating Subplant and Delay

As in Chapters 1 and 2, it is assumed that the initial function stored in the delay is the zero function. The problem addressed in this section is how to design a control scheme to minimise cost functional (3.7), assuming the connection between the delay and the subplant is inaccessible.

The first step in the design is to determine the optimal subplant input. The delay in the control ensures the subplant input for the interval $[0,\tau)$ is the zero function, in which case, the subplant output is the initial condition response

$$x(t) = e^{At} x_o \qquad 0 \leqslant t < \tau \tag{3.11}$$

The first time an input is available to the subplant is $t = \tau$, therefore, minimising cost functional (3.7) over $[\tau,\infty)$ the optimal subplant input is the solution of a LQP problem with initial conditions

$$x(\tau) = e^{A\tau} x_o \tag{3.12}$$

Section 3.1 shows the optimal subplant input is of time-invariant feedback form

$$u(t) = -Lx(t) \qquad t \geqslant \tau \tag{3.13a}$$

where

$$L = R^{-1}B^TK \tag{3.13b}$$

Remark

In the special case where a, b and consequently ℓ are scalars, the optimal subplant input for the time delay problem is revealed by considering a delay-free case with initial state

$$x(0) = e^{b\ell\tau} x_o \tag{3.14}$$

In this instance, the optimal subplant output satisfies

$$x(t) = e^{(a-\ell b)t} e^{b\ell\tau} x_o \tag{3.15a}$$

and in particular

$$x(\tau) = e^{a\tau} x_o \tag{3.15b}$$

In other words, when a control first becomes available in the time delay problem, the subplant output lies on optimal delay-free trajectory (3.15), see Figure 3.2. Bellman's Principle of Optimality, which states that any portion of an optimal trajectory is also optimal, shows the required optimal trajectory is the remainder of the delay-free trajectory. To stay on this trajectory the scalar version of control (3.13) must be implemented.

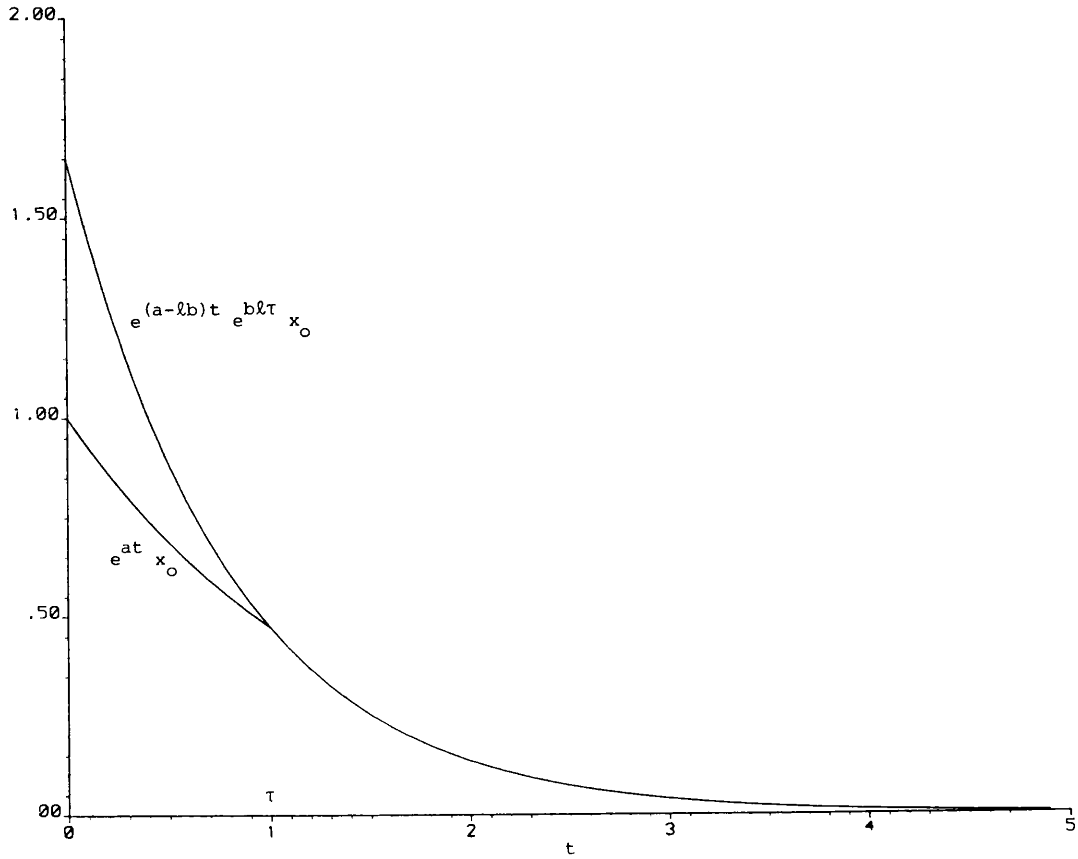


FIGURE 3.2: The Optimal Trajectory

The problem with constructing subplant input (3.13)is that the plant input must satisfy

$$w(t) = -Lx(t+\tau) \qquad t \geq 0 \tag{3.16}$$

which involves an advanced version of the state. Control scheme (1) of Figure 3.3, first presented in the given form by Marshall, Ireland and Garland (1977), realises optimal subplant input (3.13). This is a matched predictor control scheme incorporating a plant model with zero initial state and an accessible connection between subplant model and model delay. There now follows a mathematical analysis of Control scheme (1) illustrating how it achieves optimal input (3.13).

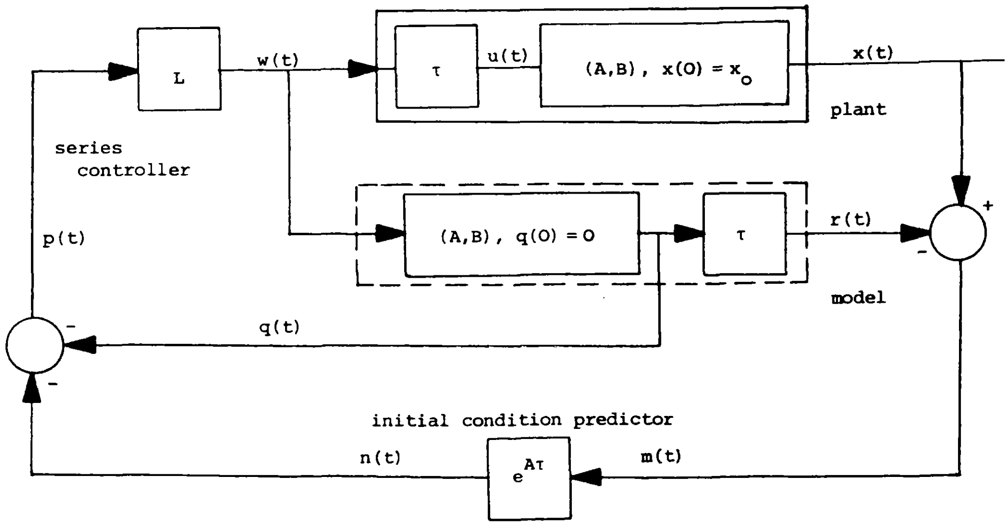


FIGURE 3.3: Control Scheme (1)

The subplant output is that of a linear system

$$x(t) = e^{At}x_o + \int_o^t e^{A(t-\sigma)}Bu(\sigma)\,d\sigma \tag{3.17}$$

As the model contains the linear system, with zero initial state, its delay-free output is given by

$$q(t) = \int_o^t e^{A(t-\sigma)}Bw(\sigma)\,d\sigma \tag{3.18}$$

The model input is an advanced version of the subplant input, therefore (3.18) may be expressed as

$$q(t) = \int_o^t e^{A(t-\sigma)}Bu(\sigma+\tau)\,d\sigma \tag{3.19}$$

and with a change of variable (3.19) may be rewritten as

$$q(t) = \int_\tau^{t+\tau} e^{A(t+\tau-\sigma)}Bu(\sigma)\,d\sigma \tag{3.20}$$

Furthermore, as the subplant input is zero on the interval $[0,\tau)$ the lower limit of integration may be taken as zero

$$q(t) = \int_{0}^{t+\tau} e^{A(t+\tau-\sigma)} Bu(\sigma)d\sigma \qquad (3.21)$$

which shows the delay-free model output is an advanced version of the forced part of subplant output (3.17).

From (3.21), the delayed model output

$$r(t) = q(t-\tau)h(t-\tau) \qquad (3.22)$$

is the forced part of the subplant output (3.17). Subtracting (3.22) from (3.17) shows the term entering the outer feedback loop is the subplant initial condition response

$$m(t) = x(t) - r(t) = e^{At}x_{o} \qquad (3.23)$$

The initial condition predictor operates on m(t) to produce an advanced version of the subplant initial condition response

$$n(t) = e^{A\tau}m(t) = e^{A(t+\tau)}x_{o} \qquad (3.24)$$

which is then added to (3.21), the advanced version of the forced response, to produce an advanced version of the subplant output

$$p(t) = -(n(t) + q(t)) = -x(t+\tau) \qquad (3.25)$$

This is operated on by the series controller

$$w(t) = -Lx(t+\tau) \qquad (3.26)$$

which, with the zero initial function in the delay, yields the optimal subplant input

$$u(t) = -Lx(t)h(t-\tau) \qquad (3.27)$$

where h(t) is the Heaviside step function.

Implementing optimal control (3.27), subplant equation (3.6) takes the form

$$\dot{x} = \{A - BLh(t-\tau)\}x, \qquad x(0) = x_o \tag{3.28}$$

It can easily be verified by direct differenciation that (3.28) has the solution

$$x(t) = \begin{cases} e^{At}x_o & 0 \le t < \tau \\ \\ e^{(A-BL)(t-\tau)}e^{A\tau}x_o & t \ge \tau \end{cases} \tag{3.29}$$

in which case (3.27) shows the optimal subplant input is given by

$$u(t) = \begin{cases} 0 & 0 \le t < \tau \\ \\ -Le^{(A-BL)(t-\tau)}e^{A\tau}x_o & t \ge \tau \end{cases} \tag{3.30}$$

In this work, a predictor control scheme is defined as stable if it produces a bounded output and the subplant is stable if all its eigenvalues lie in the left-half plane. Control scheme (1) is known to be stable irrespective of the stability of the subplant, as its output optimises a LQP problem. In the first-order case when A,B and L are replaced by scalars a,b and ℓ respectively, the above two definitions take the following form; a predictor control scheme is stable if $\exists M \in \mathbb{R}$ s.t. $\forall t \ge 0$ $|x(t)| \le M$, and the subplant is stable if $a < 0$.

In the first-order case (3.29) can be obtained in further detail. The parameter ℓ is given by

$$\ell = \frac{bk}{r} \tag{3.31}$$

where k is the positive solution of

$$\frac{b^2}{r} k^2 - 2ak - q = 0 \tag{3.32}$$

the scalar version of algebraic Riccati equation (3.9).

Therefore,

$$\ell = \frac{a + \sqrt{a^2 + b^2 q/r}}{b} \tag{3.33}$$

and

$$a - \ell b = -\sqrt{a^2 + b^2 q/r} \tag{3.34}$$

Substituting (3.34) into (3.29) the optimal subplant output in the scalar case is given by

$$x(t) = \begin{cases} e^{at} x_o & 0 \le t < \tau \\ e^{(-\sqrt{a^2 + b^2 q/r})(t-\tau)} e^{a\tau} x_o & t \ge \tau \end{cases} \tag{3.35}$$

which is clearly bounded, as after time $t = \tau$ it decays exponentially to zero, whatever the sign of parameter a. This shows explicitly that Control scheme (1) is stable, irrespective of the stability of the scalar subplant.

The drawback of Control scheme (1) is the potential for mismatch, the effects of which on stability and performance are considered in Chapters 4 and 5. Before this, Section 3.3 develops the time-varying extension of Control scheme (1).

3.3: An Optimal Control Scheme for Time-Varying Subplants

Subplant (3.1) is considered to be part of a plant with delay τ in control, and it is required to design a control scheme to minimise cost function (3.2). The assumption of zero initial function in the delay and the inaccessibility of the connection between delay and subplant also apply in this instance. The analysis involved in determining the optimal control is analogous to the time-invariant case.

The delay in control ensures the subplant input for the interval $[t_o, t_o+\tau)$ is the zero function, in which case, the subplant output is the initial condition response

$$x(t) = \Phi(t,t_o)x_o \qquad t_o \leq t < t_o + \tau \tag{3.36}$$

where $\Phi(t,t_o)$ is the transition matrix associated with $A(t)$. The first time an input is available to the subplant is $t = t_o+\tau$, therefore minimising cost functional (3.2) over $[t_o+\tau, t_f]$, the optimal subplant input is the solution of a LQP problem with initial conditions

$$x(t_o + \tau) = \Phi(t_o + \tau, t_o)x_o \tag{3.37}$$

Section 3.1 shows the optimal subplant input is of time-varying feedback form

$$u(t) = -L(t)x(t) \qquad t_o + \tau \leq t \leq t_f \tag{3.38a}$$

where

$$L(t) = R^{-1}(t)B^T(t)K(t) \qquad t_o + \tau \leq t \leq t_f \tag{3.38b}$$

Again the difficulty is that the required plant input

$$w(t) = -L(t+\tau)x(t+\tau) \qquad t_o \leq t \leq t_f \tag{3.39}$$

has an advanced version of the state as a factor. Control scheme (2) of Figure 3.4 realises optimal subplant input (3.38).

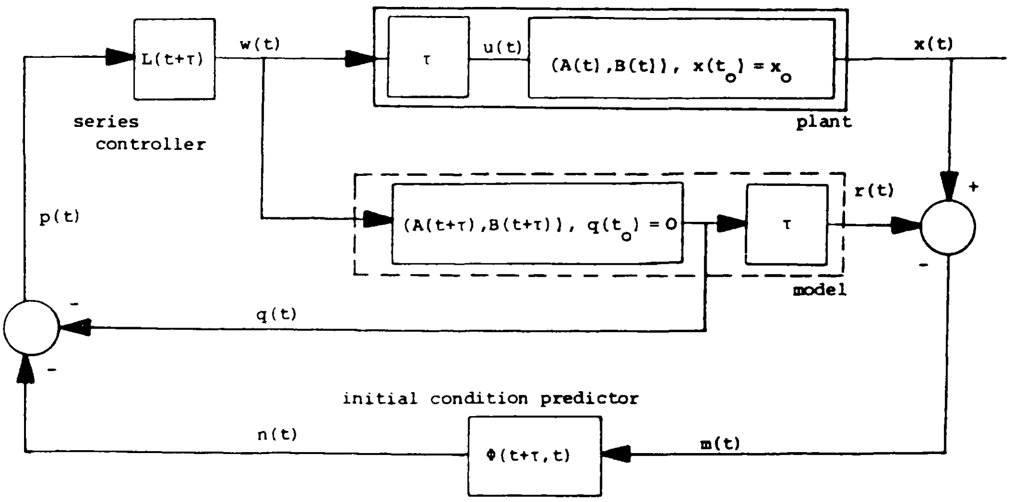


FIGURE 3.4: Control Scheme (2)

Predictor Control scheme (2) incorporates a time-varying model with an accessible connection between model subplant and model delay. The model subplant is an advanced version of the actual subplant with zero initial state. As A(t),B(t) and L(t) are only defined for $t \in [t_o, t_f]$, the choice of model subplant, initial condition predictor and series controller are undefined for $t \in (t_f - \tau, t_f]$. However, an arbitary definition may be made as they do not affect the cost functional on this interval. For example, at time $t = t_f - \tau$ Control scheme (2) produces $w(t_f - \tau) = -L(t_f)x(t_f)$ which is implemented τ seconds later as $u(t_f) = -L(t_f)x(t_f)$. In other words, cost functional (3.2) is optimised over $[t_o + \tau, t_f]$ by Control scheme (2) which effectively operates on $[t_o, t_f - \tau]$. The following is a mathematical analysis of Control scheme (2) showing how it achieves optimal subplant input (3.38).

The subplant output is that of a time-varying linear system

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma \tag{3.40}$$

As the model contains an advanced version of the linear system, with zero initial state, its delay-free output is given by

$$q(t) = \int_{t_o}^{t} \Phi(t+\tau,\ \sigma+\tau)B(\sigma+\tau)w(\sigma)d\sigma \tag{3.41}$$

The model input is an advanced version of the subplant input, therefore (3.41) can be expressed as

$$q(t) = \int_{t_o}^{t} \Phi(t+\tau,\ \sigma+\tau)B(\sigma+\tau)u(\sigma+\tau)d\sigma \tag{3.42}$$

and with a change of variable (3.42) may be rewritten as

$$q(t) = \int_{t_o+\tau}^{t+\tau} \Phi(t+\tau,\sigma)B(\sigma)u(\sigma)d\sigma \tag{3.43}$$

Furthermore, as the subplant input is zero on the interval $[t_o,t_o+\tau)$ the lower limit of integration may be taken as t_o

$$q(t) = \int_{t_o}^{t+\tau} \Phi(t+\tau,\sigma)B(\sigma)u(\sigma)d\sigma \tag{3.44}$$

which shows the delay-free model output is an advanced version of the forced part of subplant output (3.40).

From (3.44), the delayed model output

$$r(t) = q(t-\tau)h(t-(t_o+\tau)) \tag{3.45}$$

is the forced part of subplant output (3.40). Subtracting (3.45) from (3.40) shows the term entering the outer feedback loop is the subplant initial condition response

$$m(t) = x(t) - r(t) = \Phi(t,t_o)x_o \tag{3.46}$$

The initial condition predictor operates on (3.46) to produce an advanced version of the subplant initial condition response

$$n(t) = \Phi(t+\tau,t)m(t) = \Phi(t+\tau,t_o)x_o \tag{3.47}$$

which is then added to (3.44), the advanced version of the forced response, to produce an advanced version of the subplant output

$$p(t) = -(n(t) + q(t)) = -x(t+\tau) \tag{3.48}$$

The series controller operates on (3.48)

$$w(t) = -L(t+\tau)x(t+\tau) \tag{3.49}$$

and the zero initial function in the delay yields the optimal subplant input

$$u(t) = -L(t)x(t)h(t-(t_o+\tau)) \tag{3.50}$$

Implementing optimal control (3.50), subplant equation (3.1) takes the form

$$\dot{x} = \{A(t) - B(t)L(t)h(t-(t_o+\tau))\}x, \quad x(t_o) = x_o \tag{3.51}$$

the solution of which is

$$x(t) = \begin{cases} \Phi(t,t_o)x_o & t_o \le t < t_o + \tau \\ \psi(t,t_o+\tau)\ \Phi(t_o+\tau,t_o)x_o & t_o + \tau \le t \le t_f \end{cases} \tag{3.52}$$

where $\psi(t,\sigma)$ is the transition matrix associated with (A-BL)(t). From (3.50), the corresponding optimal subplant input is given by

$$u(t) = \begin{cases} 0 & t_o \leq t < t_o + \tau \\ -L(t)\psi(t,t_o+\tau)\Phi(t_o+\tau,t_o)x_o & t_o + \tau \leq t \leq t_f \end{cases} \tag{3.53}$$

It is observed that if (A(t),B(t),L(t)) are time-invariant, equations 3.40 to 3.53 reduce to the corresponding time-invariant equations 3.17 to 3.30. The essence of Control schemes (1) and (2) is that they manufacture an advanced version of the state. Therefore, the basic structure of the control schemes is applicable in any situation where an advanced version of the state is required (Grimble, 1979, 1980; Watanabe and Ito, 1982).

Conclusions

The attractiveness of LQP problems lies in the linear, feedback form of the optimal control, which is easily calculated and implemented. When quadratic cost functionals are associated to plants with control delays, the delay-free LQP results are applied to design optimal predictor control schemes. Control scheme (1) incorporates a time-invariant subplant and optimises a time-invariant cost functional over the semi-infinite time horizon. Control scheme (2) is its time-varying extension. The essence of both control schemes is the use of a perfectly matched plant model to produce an advanced version of the state. The effects of mismatch on Control scheme (1) is the subject of the following two chapters.

Chapter 4: The Form and Properties of the Mismatched Input and Output

Chapter 4: The Form and Properties of the Mismatched Input and Output

Introduction

When a quadratic cost functional is associated with a time delay plant, a matched predictor control scheme is known to be optimal (Mee, 1973). However, it is important to determine a relationship between mismatch and degradation in performance, as this will indicate the amounts by which plant and model may differ before the performance of the control scheme becomes unacceptable. A study of the effects of mismatch on Control scheme (1) constitutes the material for Chapters 4 and 5. This chapter contains the algebraic details which form the basis of the analytical and numerical results on stability and performance presented in Chapter 5.

Section 4.1 derives a matrix integral equation satisfied by the subplant input in the presence of mismatch in both the subplant and the plant delay. The solution of the equation is straightforward in the first-order case. Sections 4.2 and 4.3 describe the subplant input and output when mismatch is in the subplant parameters and Section 4.4 describes these functions for mismatch in plant delay. For each type of mismatch, three properties of the subplant input and output are illustrated. Two of these properties involve reduction; to the matched forms when mismatch is absent and to the anticipated forms on the time interval $[0,\tau]$. The third property is that of continuity. The mismatched subplant inputs are shown to be continuous for $t \neq \tau$ and the mismatched subplant outputs for $t \geqslant 0$. The chapter commences with a mathematical analysis of mismatched Control scheme (1).

4.1: A Mathematical Analysis of Mismatched Control Scheme (1)

Control scheme (1), of Figure 3.3, incorporates a plant comprising time-invariant linear system (3.6) together with a control time delay. When matched, Control scheme (1) optimises time-invariant quadratic cost functional (3.7) over the semi-infinite time horizon. However, in most practical situations, the plant will not be known exactly and mismatch will result. As in Chapters 1 and 2, it is assumed that the model is a linear system of the correct order, with a series delay, but that precise parameter values may be unknown. To study this situation, symbols A_o, B_o and τ_o are introduced to represent the possibly mismatched models of A, B and τ respectively.

As the plant is unknown, it is only realistic to calculate the initial condition predictor and the series controller from the model values. In the sequel, subscripts 1, 2 and 3 on any parameter indicate that its calculation is based on model values A_o, B_o and τ_o respectively. Figure 4.1 is a generalisation of Figure 3.3, being mismatched Control scheme (1).

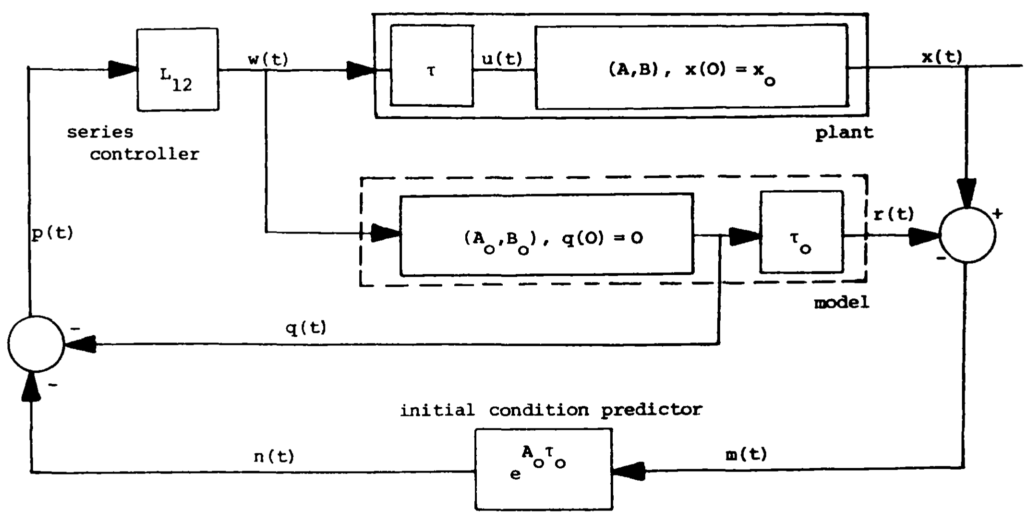


Figure 4.1: Mismatched Control Scheme (1)

A mathematical analysis of mismatched Control scheme (1) is now undertaken, and as in the matched case of Chapter 3, this results in an equation satisfied by the subplant input. In the first-order case, the Laplace transforms of the mismatched subplant input and output are easily determined from this equation.

The subplant output is that of a linear system

$$x(t) = e^{At} x_o + \int_o^t e^{A(t-\sigma)} Bu(\sigma)d\sigma \tag{4.1}$$

As the model contains the mismatched linear system, with zero initial state, the delay-free model output is given by

$$\begin{aligned} q(t) &= \int_o^t e^{A_o(t-\sigma)} B_o w(\sigma)d\sigma \\ &= \int_o^t e^{A_o(t-\sigma)} B_o u(\sigma+\tau)d\sigma \\ &= \int_o^{t+\tau} e^{A_o(t+\tau-\sigma)} B_o u(\sigma)d\sigma \end{aligned} \tag{4.2}$$

From (4.2), the delayed model output is

$$r(t) = \int_o^{t+\tau-\tau_o} e^{A_o(t+\tau-\tau_o-\sigma)} B_o u(\sigma)d\sigma \tag{4.3}$$

and subtracting (4.3) from (4.1) shows the term entering the outer feedback loop is given by

$$m(t) = x(t) - \int_o^{t+\tau-\tau_o} e^{A_o(t+\tau-\tau_o-\sigma)} B_o u(\sigma)d\sigma \tag{4.4}$$

The initial condition predictor operates on (4.4) to produce

$$n(t) = e^{A_o \tau_o} x(t) - \int_o^{t+\tau-\tau_o} e^{A_o(t+\tau-\sigma)} B_o u(\sigma) d\sigma \qquad (4.5)$$

and as q(t) and the integral term of n(t) have the same integrand, the addition of (4.2) and (4.5) results in

$$p(t) = -(e^{A_o \tau_o} x(t) + \int_{t+\tau-\tau_o}^{t+\tau} e^{A_o(t+\tau-\sigma)} B_o u(\sigma) d\sigma) \qquad (4.6)$$

The series controller operates on (4.6)

$$u(t+\tau) = L_{12}\, p(t) \qquad (4.7)$$

and the zero initial function in the delay reveals the equation satisfied by the mismatched subplant input

$$u(t) = - L_{12} \left\{ e^{A_o \tau_o} (e^{A(t-\tau)} x_o h(t-\tau) + \int_o^{t-\tau} e^{A(t-\tau-\sigma)} Bu(\sigma) d\sigma) \right.$$

$$\left. + \int_{t-\tau_o}^{t} e^{A_o(t-\sigma)} B_o u(\sigma) d\sigma \right\} \qquad (4.8)$$

It is now observed that (4.8) reduces to the desired form when plant and model are matched. When $A_o = A$, $B_o = B$, $\tau_o = \tau$ and consequently $L_{12} = L$, (4.8) satisfies

$$u(t) = - L \left\{ e^{A\tau} (e^{A(t-\tau)} x_o h(t-\tau) + \int_o^{t-\tau} e^{A(t-\tau-\sigma)} Bu(\sigma) d\sigma) \right.$$

$$\left. + \int_{t-\tau}^{t} e^{A(t-\sigma)} Bu(\sigma) d\sigma \right\}$$

$$= - L \left\{ e^{At} x_o\, h(t-\tau) + \int_o^{t-\tau} e^{A(t-\sigma)} Bu(\sigma) d\sigma + \int_{t-\tau}^{t} e^{A(t-\sigma)} Bu(\sigma) d\sigma \right\}$$

$$= - L \left\{ e^{At} x_o \, h(t-\tau) + \int_o^t e^{A(t-\sigma)} Bu(\sigma) d\sigma \right\}$$

$$= - Lx(t) \, h(t-\tau) \qquad (4.9)$$

Therefore, when mismatch is absent, the equation satisfied by the mismatched subplant input reduces to that satisfied by the matched subplant input.

It is now required to solve equation (4.8) to determine explicit formulae for the mismatched subplant input and output. This is straightforward in the first-order case where A, B, A_o and B_o are replaced by the scalar quantities a, b, a_o and b_o respectively. The Laplace transforms of the mismatched subplant input and output are determined, and particular forms of these are then inverted in the subsequent sections of Chapter 4.

Writing the integrals of (4.8) as convolutions, the first-order version becomes

$$u(t) = - \ell_{12} \left\{ e^{a_o \tau_o} \left(e^{a(t-\tau)} x_o \, h(t-\tau) + \int_o^{t-\tau} e^{a(t-\tau-\sigma)} bu(\sigma) d\sigma \right) + \int_o^t e^{a_o(t-\sigma)} b_o u(\sigma) d\sigma - e^{a_o \tau_o} \int_o^{t-\tau_o} e^{a_o(t-\tau_o-\sigma)} b_o u(\sigma) d\sigma \right\} \qquad (4.10)$$

Taking Laplace transforms throughout (4.10)

$$\bar{u}(s) = \frac{-\ell_{12} e^{a_o \tau_o} e^{-s\tau}}{s - a} - \frac{\ell_{12} e^{a_o \tau_o} be^{-s\tau} \bar{u}(s)}{s - a} - \frac{\ell_{12} b_o \bar{u}(s)}{s - a_o} + \frac{\ell_{12} e^{a_o \tau_o} b_o e^{-s\tau_o} \bar{u}(s)}{s - a_o} \qquad (4.11)$$

and rearranging the terms of (4.11) yields

$$\bar{u}(s) = \frac{-\ell_{12} e^{a_o \tau_o} x_o (s-a_o) e^{-s\tau}}{(s-a)(s-(a_o-\ell_{12}b_o)) - \ell_{12} e^{a_o \tau_o} ((s-a)b_o e^{-s(\tau_o-\tau)} - (s-a_o)b) e^{-s\tau}} \tag{4.12}$$

Introducing new notation to simplify (4.12), the Laplace transform of the subplant input in the presence of mismatch in a, b and τ simultaneously is given by

$$\bar{u}(s) = \frac{-k_{123}\, x_o (s-a_o) e^{-s\tau}}{(s-a)(s-c_{12}) - k_{123}\, d_{123}(s) e^{-s\tau}} \tag{4.13a}$$

where

$$c_{12} = a_o - \ell_{12} b_o, \quad k_{123} = \ell_{12} e^{a_o \tau_o}, \quad \ell_{12} = \frac{a_o + \sqrt{a_o^{\,2} + b_o^2 q/r}}{b_o} \tag{4.13b}$$

and

$$d_{123}(s) = (s-a) b_o e^{-s(\tau_o - \tau)} - (s-a_o) b \tag{4.13c}$$

As the subplant is a linear system, the Laplace transform of the subplant output for mismatch in a, b and τ simultaneously is given by

$$\bar{x}(s) = \frac{x_o}{s-a} + \frac{b\bar{u}(s)}{s-a} \tag{4.14}$$

where $\bar{u}(s)$ is given by (4.13)

Particular cases of (4.13) and (4.14), where mismatch is an individual parameters, are now inverted to discover the form and properties of the mismatched subplant input and output. Sections 4.2, 4.3 and 4.4 consider the cases of mismatch in a, b and τ respectively. In the remaining text of Chapter 4, the mismatched subplant input and output are referred to by the shorter mismatched input and output.

4.2: The Form and Properties of the Subplant Input and Output for Mismatch in a

4.2.1: The Form of the Input

For mismatch in a, the Laplace transform of the input is given by

$$\bar{u}(s) = \frac{-k_1 x_o (s-a_o) e^{-s\tau}}{(s-a)(s-c_1) - k_1 d_1 e^{-s\tau}} \tag{4.15a}$$

where

$$c_1 = a_o - \ell_1 b \ , \ k_1 = \ell_1 e^{a_o \tau} \ , \quad \ell_1 = \frac{a_o + \sqrt{a_o^{\ 2} + b^2 q/r}}{b} \tag{4.15b}$$

and

$$d_1 = b(a_o - a) \tag{4.15c}$$

The mechanics of inversion of (4.15) are as follows.

For $s \in \mathbf{C}$ with real part large enough

$$\left| \frac{k_1 \ d_1 \ e^{-s\tau}}{(s-a) \ (s-c_1)} \right| < 1 \tag{4.16}$$

and (4.15a) can be expressed as a geometric series

$$\bar{u}(s) = \frac{- k_1 x_o (s-a_o) e^{-s\tau}}{(s-a)(s-c_1)\left\{1 - \dfrac{k_1 d_1 e^{-s\tau}}{(s-a)(s-c_1)}\right\}}$$

$$= - k_1 x_o (s-a_o) \sum_{n=1}^{\infty} \frac{(k_1 d_1)^{n-1} e^{-ns\tau}}{(s-a)^n (s-c_1)^n} \qquad (4.17)$$

Now, expanding by partial fractions

$$\frac{1}{(s-a)^n (s-c_1)^n} = \sum_{i=0}^{n-1} p(n,i) \left\{ \frac{1}{(c_1-a)^{n+i}} \frac{1}{(s-c_1)^{n-i}} + \frac{1}{(a-c_1)^{n+i}} \frac{1}{(s-a)^{n-i}} \right\} \qquad (4.18)$$

where

$$p(n,o) = 1 \quad \text{and} \quad p(n,i) = (-1)^i \frac{n(n+1)\ldots(n+i-1)}{i!} \quad 1 \leqslant i \leqslant n-1 \qquad (4.19)$$

Adopting the notation

$$\mathcal{L}^{-1} \frac{(k_1 d_1)^{n-1}}{(s-a)^n (s-c_1)^n} = f_n(t) \qquad (4.20)$$

the inverse Laplace transform of (4.18) shows

$$f_n(t) = (k_1 d_1)^{n-1} \sum_{i=0}^{n-1} f_{n_i}(t) \qquad (4.21)$$

where

$$f_{n_i}(t) = p(n,i) \frac{t^{n-i-1}}{(n-i-1)!} \left\{ \frac{e^{at}}{(a-c_1)^{n+i}} + \frac{e^{c_1 t}}{(c_1-a)^{n+i}} \right\} \qquad (4.22)$$

Furthermore, substituting $t = 0$ into (4.21) and (4.22) reveals

$$f_n(o) = (k_1 d_1)^{n-1}\, p(n,n-1)\left\{\frac{1}{(a-c_1)^{2n-1}} + \frac{1}{(c_1-a)^{2n-1}}\right\} = o \qquad (4.23)$$

in which case

$$\mathcal{L}^{-1}\ \frac{s(k_1 d_1)^{n-1}}{(s-a)^n (s-c_1)^n} = f_n'(t) \qquad (4.24)$$

Applying these results, the inverse Laplace transform of (4.17) shows the input for mismatch in a is given by

$$u(t) = -\,k_1\, x_o \sum_{n=1}^{\infty} \{f_n'\,(t-n\tau) - a_o\, f_n(t-n\tau)\}\, h(t-n\tau) \qquad (4.25a)$$

where

$$f_n(t) = (k_1 d_1)^{n-1} \sum_{i=o}^{n-1} f_{n_i}(t) \qquad (4.25b)$$

$$f_{n_i}(t) = p(n,i)\ \frac{t^{n-i-1}}{(n-i-1)!}\left\{\frac{e^{at}}{(a-c_1)^{n+i}} + \frac{e^{c_1 t}}{(c_1-a)^{n+i}}\right\} \qquad (4.25c)$$

$$p(n,o) = 1\ ,\ p(n,i) = \frac{(-1)^i n(n+1)\ldots(n+i-1)}{i!}\quad 1 \leqslant i \leqslant n-1 \qquad (4.25d)$$

$$c_1 = a_o - \ell_1 b\ ,\ k_1 = \ell_1 e^{a_o\tau}\ ,\ \ell_1 = \frac{a_o + \sqrt{a_o^{\,2} + b^2\, q/r}}{b} \qquad (4.25e)$$

and

$$d_1 = b(a_o - a) \qquad (4.25f)$$

The mismatched input is an infinite series consisting of terms with exponential factors and factors which are powers of t . The n^{th} term contains delay $n\tau$, that is, it does not contribute to the series before time $t = n\tau$. The remainder of Section 4.2 includes particular properties of the input for mismatch in a. The first of these is the reduction of (4.25) to the matched form when mismatch is absent.

4.2.2: The Reduction of the Input to Matched Form

When plant and model match $a_o = a$ and the parameters of (4.25e) and (4.25f) satisfy

$$d_1 = d = 0\ ,\quad \ell_1 = \ell\ ,\quad k_1 = k\ ,\quad c_1 = c \tag{4.26}$$

The presence of zero valued d_1 in (4.25b) ensures

$$f_n(t) = 0\ ,\quad f_n'(t) = 0 \qquad n = 2,3,\ldots \tag{4.27}$$

in which case (4.25a) reduces to

$$u(t) = -\,kx_o\ \{f_1'(t-\tau) - af_1(t-\tau)\}\,h(t-\tau) \tag{4.28}$$

Examining the terms of (4.28) in detail

$$\begin{aligned} f_1(t-\tau) &= f_{1_o}(t-\tau) \\ &= \frac{e^{a(t-\tau)}}{a-c} + \frac{e^{c(t-\tau)}}{c-a} \\ &= \frac{e^{a(t-\tau)} - e^{(a-\ell b)(t-\tau)}}{\ell b} \end{aligned} \tag{4.29}$$

and

$$f_1'(t-\tau) = \frac{ae^{a(t-\tau)} - (a-\ell b)e^{(a-\ell b)(t-\tau)}}{\ell b} \tag{4.30}$$

Combining (4.29) and (4.30)

$$f_1'(t-\tau) - af_1(t-\tau) = e^{(a-\ell b)(t-\tau)} \tag{4.31}$$

Substituting (4.31) into (4.28) mismatched input (4.25a) reduces to

$$u(t) = - \ell e^{(a-\ell b)(t-\tau)} e^{a\tau} x_o h(t-\tau) \tag{4.32}$$

the desired first-order version of matched input (3.30). Section 4.2.3 now observes that the mismatched input reduces to the anticipated form over the interval $[0,\tau]$.

4.2.3: The Input Over $[0,\tau]$

When $t < \tau$, by the definition of the Heaviside step function

$$h(t-n\tau) = 0 \qquad n = 1,2,... \tag{4.33}$$

and (4.25a) reduces to

$$u(t) = 0 \qquad 0 \leqslant t < \tau \tag{4.34}$$

It therefore remains to consider the case of $t = \tau$. In this instance

$$h(t-n\tau) = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,... \end{cases} \tag{4.35}$$

which ensures (4.25a) satisfies

$$u(\tau) = - k_1 x_o \{f_1'(0) - a_o f_1(0)\} \tag{4.36}$$

Examining the terms of (4.36) in detail

$$f_1(t) = f_{1_o}(t)$$

$$= \frac{e^{at}}{a-c_1} + \frac{e^{c_1 t}}{c_1-a} \tag{4.37}$$

and

$$f_1'(t) = \frac{ae^{at}}{a-c_1} + \frac{c_1 e^{c_1 t}}{c_1-a} \tag{4.38}$$

in which case, when $t = 0$

$$f_1(0) = \frac{1}{a-c_1} + \frac{1}{c_1-a} = 0 \tag{4.39}$$

and

$$f_1'(0) = \frac{a}{a-c_1} + \frac{c_1}{c_1-a} = 1 \tag{4.40}$$

Substituting (4.39) and (4.40) into (4.36) shows that for the case $t = \tau$ (4.25a) reduces to

$$u(\tau) = - k_1 x_o \tag{4.41}$$

Combining (4.34) and (4.41), the input for mismatch in a satisfies

$$u(t) = \begin{cases} 0 & 0 \leqslant t < \tau \\ -\ell e^{a_o \tau} x_o & t = \tau \end{cases} \tag{4.42}$$

which is the expected form over $[0,\tau]$. The final property to be considered is that of continuity. The input for mismatch in a is clearly continuous for $t < \tau$ and discontinuous at $t = \tau$. Section 4.2.4 shows mismatched input (4.25) is continuous for $t > \tau$.

4.2.4: The Continuity of the Input

To prove continuity for $t > \tau$ it is required to show

$$f'_n(0) - a_o f_n(0) = 0 \qquad n = 2,3,\ldots \tag{4.43}$$

This ensures the n^{th} term $\{f'_n(t-n\tau) - a_o f_n(t-n\tau)\}h(t-n\tau)$ is zero when first contributing to (4.25) at time $t = n\tau$. Consequently, the mismatched input changes smoothly between adjoining time intervals and is continuous for $t \neq \tau$.

Substituting $t = 0$ into (4.25b) and (4.25c)

$$f_n(0) = (k_1 d_1)^{n-1} \sum_{i=0}^{n-1} f_{n_i}(0) \tag{4.44}$$

where

$$f_{n_i}(0) = 0 \qquad 0 \leqslant i \leqslant n-2 \tag{4.45}$$

and

$$f_{n_{n-1}}(0) = p(n,n-1) \left\{ \frac{1}{(a-c_1)^{2n-1}} + \frac{1}{(c_1-a)^{2n-1}} \right\} = 0 \tag{4.46}$$

Substituting (4.45) and (4.46) into (4.44)

$$f_n(0) = 0 \qquad n = 1,2,\ldots \tag{4.47}$$

From (4.25b) and (4.25c), the derivative of $f_n(t)$ is given by

$$f'_n(t) = (k_1 d_1)^{n-1} \sum_{i=0}^{n-1} f'_{n_i}(t) \tag{4.48}$$

where for $0 \leqslant i \leqslant n-2$, $n = 2,3,\ldots$

$$f'_{n_i}(t) = p(n,i)\left\{\frac{t^{n-i-1}}{(n-i-1)!}\left\{\frac{ae^{at}}{(a-c_1)^{n+i}} + \frac{c_1e^{c_1t}}{(c_1-a)^{n+i}}\right\}\right.$$

$$\left. + \frac{t^{n-i-2}}{(n-i-2)!}\left\{\frac{e^{at}}{(a-c_1)^{n+i}} + \frac{e^{c_1t}}{(c_1-a)^{n+i}}\right\}\right\} \qquad (4.49)$$

and for $n = 1,2,\ldots$

$$f'_{n_{n-1}}(t) = p(n,n-1)\left\{\frac{ae^{at}}{(a-c_1)^{2n-1}} + \frac{c_1e^{c_1t}}{(c_1-a)^{2n-1}}\right\} \qquad (4.50)$$

Substituting $t = 0$ into (4.49)

$$f'_{n_i}(0) = 0 \quad 0 \leqslant i \leqslant n-3 \quad n = 3,4,\ldots \qquad (4.51)$$

and for $n = 2,3,\ldots$

$$f'_{n_{n-2}}(0) = 2p(n,n-2)\frac{1}{(a-c_1)^{2n-2}} \qquad (4.52)$$

Substituting $t = 0$ into (4.50)

$$f'_{n_{n-1}}(0) = p(n,n-1)\frac{1}{(a-c_1)^{2n-2}} \quad n = 1,2,\ldots \qquad (4.53)$$

Adding (4.52) and (4.53) shows that for $n = 2,3,\ldots$

$$f'_{n_{n-2}}(0) + f'_{n_{n-1}}(0) = 2p(n,n-2) + p(n,n-1)\frac{1}{(a-c_1)^{2n-2}} = 0 \qquad (4.54)$$

Substituting (4.51) and (4.54) into (4.48)

$$f_n'(0) = 0 \qquad n = 2,3,\ldots \tag{4.55}$$

Combining (4.47) and (4.55) gives (4.43) as required, and the input for mismatch in a is continuous for $t \neq \tau$. Attention is now turned to the form and properties of the output for mismatch in a.

4.2.5: The Form and Properties of the Output

For mismatch in a, the Laplace transform of the output is given by

$$\bar{x}(s) = \frac{x_o}{s-a} - \frac{bk_1x_o(s-a_o)e^{-s\tau}}{(s-a)((s-a)(s-c_1)-k_1d_1e^{-s\tau})} \tag{4.56a}$$

where

$$c_1 = a_o - \ell_1 b \;, \quad k_1 = \ell_1 e^{a_o\tau} \;, \quad \ell_1 = \frac{a_o+\sqrt{a_o^{\,2}+b^2q/r}}{b} \tag{4.56b}$$

and

$$d = b(a_o-a) \tag{4.56c}$$

The algebraic details associated with the form and properties of the output for mismatch in a, are similar to those presented for the corresponding input. Consequently, they are included as Section A1 of the Appendix. Inverting (4.56) the output for mismatch in a is given by

$$x(t) = e^{at}x_o - bk_1x_o \sum_{n=1}^{\infty} \{g_n'(t-n\tau) - a_o g_n(t-n\tau)\}\, h(t-n\tau) \tag{4.57a}$$

where

$$g_n(t) = (k_1d_1)^{n-1} \Big(\sum_{i=o}^{n} \hat{g}_{n_i}(t) + \sum_{i=o}^{n-1} \tilde{g}_{n_i}(t) \Big) \tag{4.57b}$$

$$\hat{g}_{n_i}(t) = p(n,i) \frac{1}{(a-c_1)^{n+i}} \frac{t^{n-i}}{(n-i)!} e^{at} \tag{4.57c}$$

$$\tilde{g}_{n_i}(t) = p(n+1,i) \frac{1}{(c_1-a)^{n+i+1}} \frac{t^{n-i-1}}{(n-i-1)!} e^{c_1t} \tag{4.57d}$$

$$p(m,o) = 1, \quad p(m,i) = \frac{(-1)^i m(m+1)\ldots(m+i-1)}{i!} \qquad 1 \leqslant i \leqslant m-1 \tag{4.57e}$$

$$c_1 = a_o - \ell_1 b, \quad k_1 = \ell_1 e^{a_o\tau}, \qquad \ell_1 = \frac{a_o + \sqrt{a_o^2+b^2q/r}}{b} \tag{4.57f}$$

and

$$d_1 = b(a_o-a) \tag{4.57g}$$

The mismatched output is also an infinite series of terms comprising exponential factors and factors which are powers of t. Again, owing to the presence of delay $n\tau$ the n^{th} term does not contribute before time $t = n\tau$. The initial condition response $e^{at}x_o$ appears explicitly in the formula and (4.57) reduces to this on the interval $[0,\tau]$. In the absence of mismatch, (4.57) is seen to reduce to its matched form, the first-order version of (3.29). Furthermore, as a result of the mismatched input being continuous for $t \neq \tau$, it is easily deduced that the mismatched output is continuous for $t \geqslant 0$. Section 4.3 now considers the form and properties of the input and output when mismatch is in the other subplant parameter.

4.3: The Form and Properties of the Subplant Input and Output for Mismatch in b

For mismatch in b , the Laplace transform of the input is given by

$$\bar{u}(s) = \frac{-k_2 x_o e^{-s\tau}}{(s-c_2) - k_2 d_2 e^{-s\tau}} \tag{4.58a}$$

where

$$c_2 = a - \ell_2 b_o \ , \quad k_2 = \ell_2 e^{a\tau} \ , \quad \ell_2 = \frac{a + \sqrt{a^2+b_o{}^2 q/r}}{b_o} \tag{4.58b}$$

and

$$d_2 = b_o - b \tag{4.58c}$$

The similarity with the case of mismatch in a , allows the algebraic details for mismatch in b to be relegated to Section A2 of the Appendix. Inverting (4.58) the input for mismatch in b is given by

$$u(t) = -k_2 x_o \sum_{n=1}^{\infty} f_n(t-n\tau)\, h(t-n\tau) \tag{4.59a}$$

where

$$f_n(t) = (k_2 d_2)^{n-1} \frac{t^{n-1}}{(n-1)!} e^{c_2 t} \tag{4.59b}$$

$$c_2 = a - \ell_2 b_o \ , \quad k_2 = \ell_2 e^{a\tau} \ , \quad \ell_2 = \frac{a + \sqrt{a^2+ b_o{}^2 q/r}}{b_o} \tag{4.59c}$$

and

$$d_2 = b_o - b \tag{4.59d}$$

The Laplace transform of the mismatched output is determined by (4.14). Inverting this, the output for mismatch in b is given by

$$x(t) = e^{at}x_o - bk_2x_o \sum_{n=1}^{\infty} g_n(t-n\tau)\ h(t-n\tau) \tag{4.60a}$$

where

$$g_n(t) = (k_2d_2)^{n-1}\left\{\frac{e^{at}}{(a-c_2)^n} + \sum_{i=o}^{n-1} g_{n_i}(t)\right\} \tag{4.60b}$$

$$g_{n_i}(t) = \frac{(-1)^i}{(c_2-a)^{i+1}}\ \frac{t^{n-i-1}}{(n-i-1)!}\ e^{c_2t} \tag{4.60c}$$

$$c_2 = a - \ell_2b_o\ ,\quad k_2 = \ell_2e^{a\tau}\ ,\quad \ell_2 = \frac{a + \sqrt{a^2+b_o^{\,2}q/r}}{b_o} \tag{4.60d}$$

and

$$d_2 = b_o - b \tag{4.60e}$$

The input and output for mismatch in b are similar to the corresponding quantities for mismatch in a. The n^{th} terms of (4.59a) and (4.60a) comprise exponential factors, power of t factors and the delay $n\tau$. However, (4.59) and (4.60) do not contain any derivative terms, as s in the numerator of (4.58a) is restricted to the delay. Both (4.59) and (4.60) reduce to their expected forms, when mismatch is absent and over the interval $[0,\tau]$. The continuity properties of (4.59) and (4.60) are as for mismatch in a. The mismatched input is continuous for $t \neq \tau$ and the mismatched output for $t \geqslant 0$. The final section of Chapter 4 considers the form and properties of the input and output for mismatch in delay.

4.4 : The Form and Properties of the Subplant Input and Output for Mismatch in Delay

4.4.1: The Form of the Input

For mismatch in delay, the Laplace transform of the input is given by

$$\bar{u}(s) = \frac{-k_3 x_o e^{-s\tau}}{(s-c)-k_3 d_3(s) e^{-s\tau}} \tag{4.61a}$$

where

$$c = a - \ell b \ , \quad k_3 = \ell e^{a\tau_o} \ , \quad \ell = \frac{a + \sqrt{a^2+b^2q/r}}{b} \tag{4.61b}$$

and

$$d_3(s) = b(e^{-s(\tau_o-\tau)} - 1) \tag{4.61c}$$

The presence of a model delay ensures the form and properties of the mismatched input and output are somewhat different from the results of Sections 4.2 and 4.3. For this reason, the algebraic details for mismatch in delay are included here. The mechanics of inversion of (4.61) are as follows. For $s \in \mathbb{C}$ with real part large enough

$$\left| \frac{k_3 b(e^{-s\tau_o} - e^{-s\tau})}{s-c} \right| < 1 \tag{4.62}$$

and (4.61a) can be expressed as a geometric series

$$\bar{u}(s) = \frac{-k_3 x_o e^{-s\tau}}{(s-c)\left\{1 - \dfrac{k_3 b(e^{-s\tau_o} - e^{-s\tau})}{s-c}\right\}}$$

$$= -k_3 x_o \sum_{n=1}^{\infty} \frac{(k_3 b)^{n-1} (e^{-s\tau_o} - e^{-s\tau})^{n-1} \; e^{-s\tau}}{(s-c)^n}$$

$$= -k_3 x_o \sum_{n=1}^{\infty} (k_3 b)^{n-1} \sum_{i=o}^{n-1} \frac{(-1)^{n-i-1}}{(n-i-1)!} \frac{(n-1)!}{i!} \; \frac{e^{-s(i\tau_o + (n-i)\tau)}}{(s-c)^n} \qquad (4.63)$$

Adopting the notation

$$\mathcal{L}^{-1} \; \frac{(k_3 b)^{n-1}}{(s-c)^n} = f_n(t) \qquad (4.64)$$

the inverse Laplace transform of (4.63) shows the input for mismatch in delay is given by

$$u(t) = -k_3 x_o \sum_{n=1}^{\infty} \sum_{i=o}^{n-1} m(n,i) f_n(t-\lambda_{n_i}) h(t-\lambda_{n_i}) \qquad (4.65a)$$

where

$$f_n(t) = (k_3 b)^{n-1} \frac{t^{n-1}}{(n-1)!} \; e^{ct} \qquad (4.65b)$$

$$m(n,i) = \frac{(-1)^{n-i-1}}{(n-i-1)!} \frac{(n-1)!}{i!} \; , \quad \lambda_{n_i} = (n-i)\tau + i\tau_o \; ; \quad 0 \leqslant i \leqslant n-1 \qquad (4.65c)$$

and

$$c = a - \ell b \; , \quad k_3 = \ell e^{a\tau_o} \; , \quad \ell = \frac{a + \sqrt{a^2+b^2 q/r}}{b} \qquad (4.65d)$$

The mismatched input is again an infinite series of terms comprising exponential factors and factors which are powers of t. However, the n^{th} term of the series now consists of a sum of n subterms, the i^{th} of which contains a delay of the form $(n-i)\tau + i\tau_o$, $0 \leqslant i \leqslant n-1$. It is now shown that (4.65) reduces to the matched form when mismatch is absent. Other properties considered are continuity and the form over interval $[0,\tau]$.

4.4.2: The Reduction of the Input to Matched Form

When plant and model match $\tau_o = \tau$ and the parameters of (4.65c) and (4.65d) satisfy

$$\lambda_{n_i} = n\tau \ , \quad 0 \leqslant i \leqslant n-1 \ , \quad k_3 = k \tag{4.66}$$

which ensures (4.65a) becomes

$$u(t) = -kx_o \sum_{n=1}^{\infty} \sum_{i=o}^{n-1} m(n,i)\ f_n(t-n\tau)\ h(t-n\tau) \tag{4.67}$$

The sum over i now involves only the terms of $m(n,i)$

$$\sum_{i=o}^{n-1} m(n,i) = \sum_{i=o}^{n-1} \frac{(n-1)!\,(-1)^{n-i-1}}{(n-i-1)!\ \ i!} = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,\ldots \end{cases} \tag{4.68}$$

and (4.67) reduces to

$$u(t) = -kx_o f_1(t-\tau)\ h(t-\tau) \tag{4.69}$$

From (4.65b) and (4.65d)

$$f_1(t-\tau) = e^{(a-\ell b)(t-\tau)} \tag{4.70}$$

Substituting (4.70) into (4.69), mismatched input (4.65a) reduces to

$$u(t) = -\ell e^{(a-\ell b)(t-\tau)} e^{a\tau} x_o \; h(t-\tau) \tag{4.71}$$

the desired first-order version of matched input (3.30). Section 4.4.3 now observes that (4.65) reduces to the anticipated form over interval $[0,\tau]$.

4.4.3: The Input over $[0,\tau]$

When $t < \tau$, by definition of the Heaviside step function

$$h(t - \lambda_{n_i}) = 0 \qquad n = 1,2,\ldots \tag{4.72}$$

and (4.65a) reduces to

$$u(t) = 0 \qquad 0 \leqslant t < \tau \tag{4.73}$$

It therefore remains to consider the case of $t = \tau$. In this instance

$$h(t - \lambda_{n_i}) = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,\ldots \end{cases} \tag{4.74}$$

which ensures (4.65a) satisfies

$$u(\tau) = -k_3 x_o m(1,0) f_1(0) \tag{4.75}$$

From (4.65b) and (4.65c)

$$f_1(0) = 1 \tag{4.76}$$

and

$$m(1,0) = 1 \tag{4.77}$$

Substituting (4.76) and (4.77) into (4.75) shows that for the case $t = \tau$ (4.65a) reduces to

$$u(\tau) = -k_3 x_o \tag{4.78}$$

Combining (4.73) and (4.78), the input for mismatch in delay satisfies

$$u(t) = \begin{cases} 0 & 0 \leqslant t < \tau \\ -\ell e^{a\tau_o} x_o & t = \tau \end{cases} \tag{4.79}$$

which is the expected form over $[0,\tau]$. It remains to discuss the property of continuity. The input for mismatch in delay is clearly continuous for $t < \tau$ and discontinuous at $t = \tau$. Section 4.4.4 shows mismatched input (4.65) is continuous for $t > \tau$.

4.4.4: The Continuity of the Input

To prove continuity for $t > \tau$ it is required to show

$$f_n(0) = 0 \qquad n = 2,3,\ldots \tag{4.80}$$

This ensures $m(n,i)f_n(t-\lambda_{n_i})h(t-\lambda_{n_i})$ is zero when first contributing to (4.65) at time $t = (n-i)\tau + i\tau_o$, $0 \leqslant i \leqslant n-1$. Consequently, the mismatched input changes smoothly between adjointing time intervals and is continuous for $t \neq \tau$.

Substituting $t = 0$ into (4.65b) shows immediately that

$$f_n(0) = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,\ldots \end{cases} \tag{4.81}$$

which satisifes (4.80) as required. Attention is now turned to the form and properties of the output for mismatch in delay.

4.4.5: The Form and Properties of the Output

For mismatch in delay, the Laplace transform of the output is given by

$$\bar{x}(s) = \frac{x_o}{s-a} - \frac{bk_3x_oe^{-s\tau}}{(s-a)\,((s-c) - k_3b(e^{-s\tau_o} - e^{-s\tau}))} \tag{4.82a}$$

where

$$c = a - \ell b\ , \quad k_3 = \ell e^{a\tau_o}\ , \quad \ell = \frac{a + \sqrt{a^2+b^2q/r}}{b} \tag{4.82b}$$

The algebraic details associated with the form and properties of the output for mismatch in delay, are similar to those presented for the corresponding input. Consequently, they are included as Section A3 of the Appendix.

Inverting (4.82) the output for mismatch in delay is given by

$$x(t) = e^{at}x_o - bk_3x_o \sum_{n=1}^{\infty} \sum_{i=o}^{n-1} m(n,i)\, g_n(t-\lambda_{n_i})h(t-\lambda_{n_i}) \tag{4.83a}$$

where

$$g_n(t) = \frac{e^{at}}{(a-c)^n} + \sum_{j=o}^{n-1} g_{n_j}(t) \tag{4.83b}$$

$$g_{n_j}(t) = \frac{(-1)^j}{(c-a)^{j+1}} \frac{t^{n-j-1}}{(n-j-1)!} e^{ct} \tag{4.83c}$$

$$m(n,i) = \frac{(-1)^{n-i-1}(n-1)!}{i!\,(n-i-1)!}, \quad \lambda_{n_i} = (n-i)\tau + i\tau_o \; ; \quad 0 \leqslant i \leqslant n-1 \tag{4.83d}$$

and

$$c = a - \ell b\, , \quad k_3 = \ell e^{a\tau_o}\, , \quad \ell = \frac{a + \sqrt{a^2+b^2q/r}}{b} \tag{4.83e}$$

The mismatched output is again an infinite series of terms comprising exponential factors and factors which are powers of t. As in the case of the input, the n^{th} term is a sum of n subterms, the i^{th} of which contains a delay of the form $(n-i)\tau + i\tau_o$, $0 \leqslant i \leqslant n-1$. The mismatched output is slightly more involved than the corresponding input in that it involves a triple rather than a double sum. The mismatched output reduces to the matched form when mismatch is absent, has the expected form over $[0,\tau]$ and is continuous for $t \geqslant 0$.

Conclusions

Control scheme (1) utilises a plant model which introduces the possibility of mismatch. A mathematical analysis of mismatched Control scheme (1) yields a matrix integral equation satisfied by the subplant input. The solution of this integral equation is straightforward in the first-order case. Expressing the integrals as convolutions, the Laplace transforms of the mismatched input and output are easily obtained. For mismatch in individual parameters, the Laplace transforms are then expressed as geometric series and inverted term by term. This has the result that both mismatched input and output are infinite series with terms consisting of delays, exponential factors and factors which are powers of t .

For mismatch in the subplant parameters, the n^{th} term of the series contains delay $n\tau$, that is, it does not contribute to the series before time $t = n\tau$. For mismatch in delay, the n^{th} term of the series consists of a sum of n subterms, the i^{th} of which contains a delay $(n-i)\tau + i\tau_o$, $0 \leqslant i \leqslant n-1$. The mismatched inputs and outputs reduce to their matched forms when mismatch is absent and to their anticipated forms over $[0,\tau]$. Furthermore, the mismatched inputs and outputs are continuous for $t \neq \tau$ and $t \geqslant 0$ respectively.

Chapter 5 comprises a numerical and analytical examination of the expressions for the mismatched input and output. This determines how mismatch affects the stability and performance of Control scheme (1) for a representative selection of first-order subplants. A summary of the work of Chapters 4 and 5 is contained in Hocken and Marshall (1983).

Chapter 5: The Effects of Mismatch on Stability and Performance — Page

Chapter 5: The Effects of Mismatch on Stability and Performance

Introduction

Chapter 5 comprises a numerical and analytical examination of the expressions of Chapter 4 for the mismatched input and output. This determines how mismatch affects the stability and performance of Control scheme (1) for a representative selection of first-order subplants. It is known from Chapter 3 that matched Control scheme (1) is stable regardless of the stability of the subplant, as its output optimises a LQP problem.

Sections 5.1, 5.2 and 5.3 consider the stability of Control scheme (1) for mismatch in a, b and τ respectively. For each type of mismatch, the cases of stable and unstable subplants are treated in distinct subsections. Intervals of model parameter values are found for which mismatched Control scheme (1) is stable. Similar intervals of stability have been studied by the authors listed in Section 1.4. Some general results on mismatch and stability are collected together in Section 5.4. Finally, Section 5.5 covers the performance of Control scheme (1) in the presence of mismatch. Curves representing the variation of the quadratic cost functional with model parameters are presented. From these curves it is possible to deduce whether, in the absence of the matched values, underestimation or overestimation of particular parameters is preferable.

For mismatch in a and mismatch in delay, the results of this chapter are obtained when Control scheme (1) incorporates either plant $(a,b,\tau)=(-1,3,1)$ or $(a,b,\tau)=(1,3,1)$. For mismatch in b the selection is between $(a,b,\tau)=(-1,1,1)$ and $(a,b,\tau)=(1,1,1)$. For each case of mismatch, the choice of plant depends on whether a stable or an unstable subplant is being considered. In every case, the cost functional parameters are chosen to be $(q,r)=(1,1)$.

For each form of the mismatched input and output the role of the initial state of the subplant is as a scaling factor. As the initial state has no bearing on the stability and performance of mismatched Control scheme (1), it may be chosen for convenience. When the subplant is stable a suitable choice for the initial state is $x_o = 1.0$. In the remaining text of Chapter 5, the mismatched input and output are referred to by the shorter input and output. Furthermore, no confusion should arise from referring to mismatched Control scheme (1) simply as Control scheme (1). The next section of Chapter 5 examines the stability of Control scheme (1) for mismatch in a.

5.1: The Stability of Control Scheme (1) for Mismatch in a

5.1.1: A Stable Subplant

Control scheme (1) is examined when it incorporates plant $(a,b,\tau)=(-1,3,1)$. The first cases of mismatch to be considered constitute an underestimation of $a=-1$. Figure 5.1 shows the change from the exponentially decaying matched input and output when $a_o=-1$ is replaced by the underestimation $a_o=-2$. For a stable subplant, as a_o becomes large and negative input (4.25) tends to the zero function and the corresponding output (4.57) approaches the initial condition response.

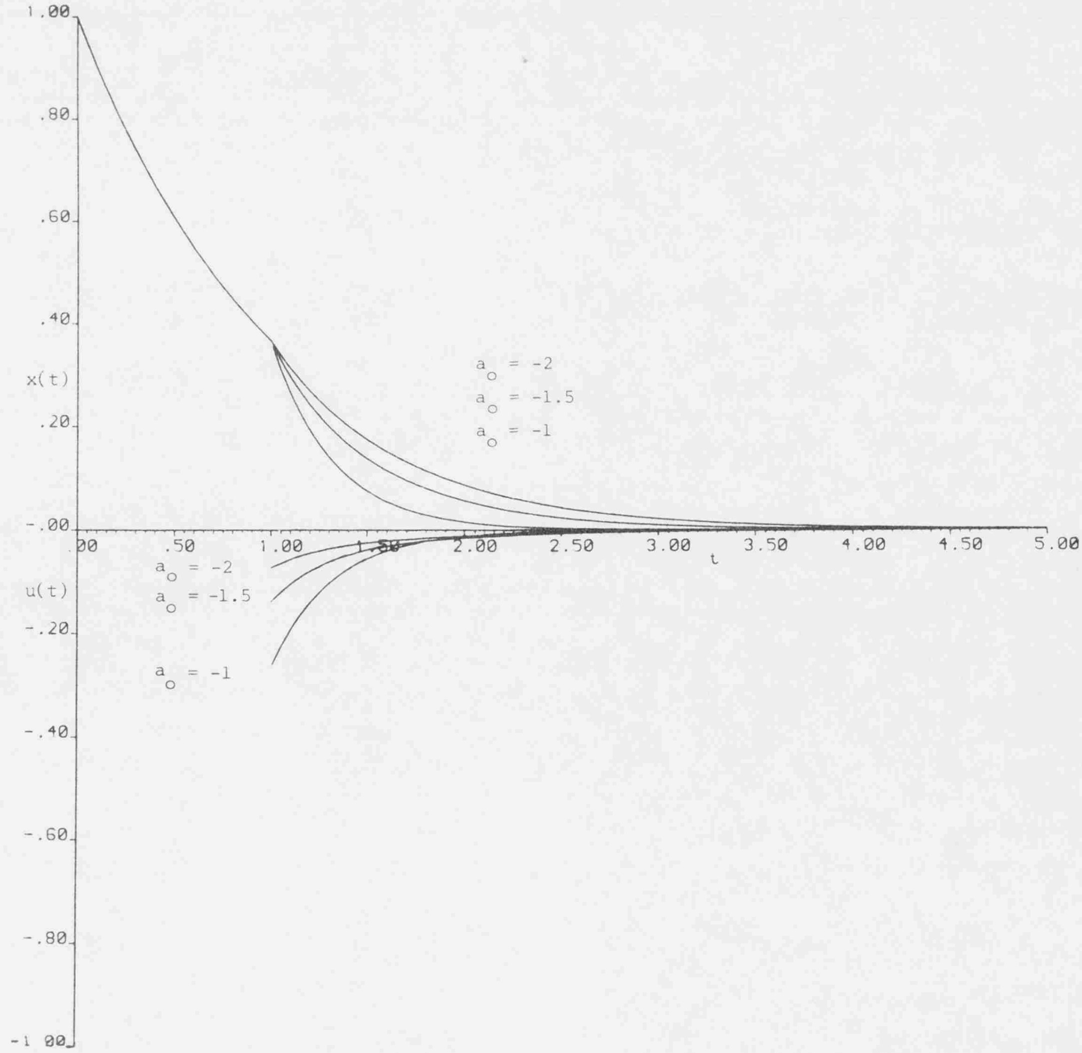


FIGURE 5.1: The Input and Output of a Stable Subplant for Underestimation of a

The forms of input (4.25) and output (4.57) for overestimates of $a = -1$ are shown in Figure 5.2. To aid comparison, the matched case is repeated in Figure 5.2a. As a_o is increased above the matched value, Figure 5.2b shows the input and output become increasingly oscillatory. Eventually subplant parameter $a = -1$ is overestimated to the extent that the subplant model becomes unstable, as in Figure 5.2c. However, the input and output continue to converge to zero, as they do in Figure 5.2d, Control scheme (1) remaining stable. When the model parameter is further increased to $a_o = 0.97$, Figure 5.2e shows the input and output are bounded but non-convergent. Any further increase in a_o results in an unstable control scheme, as seen in Figure 5.2f.

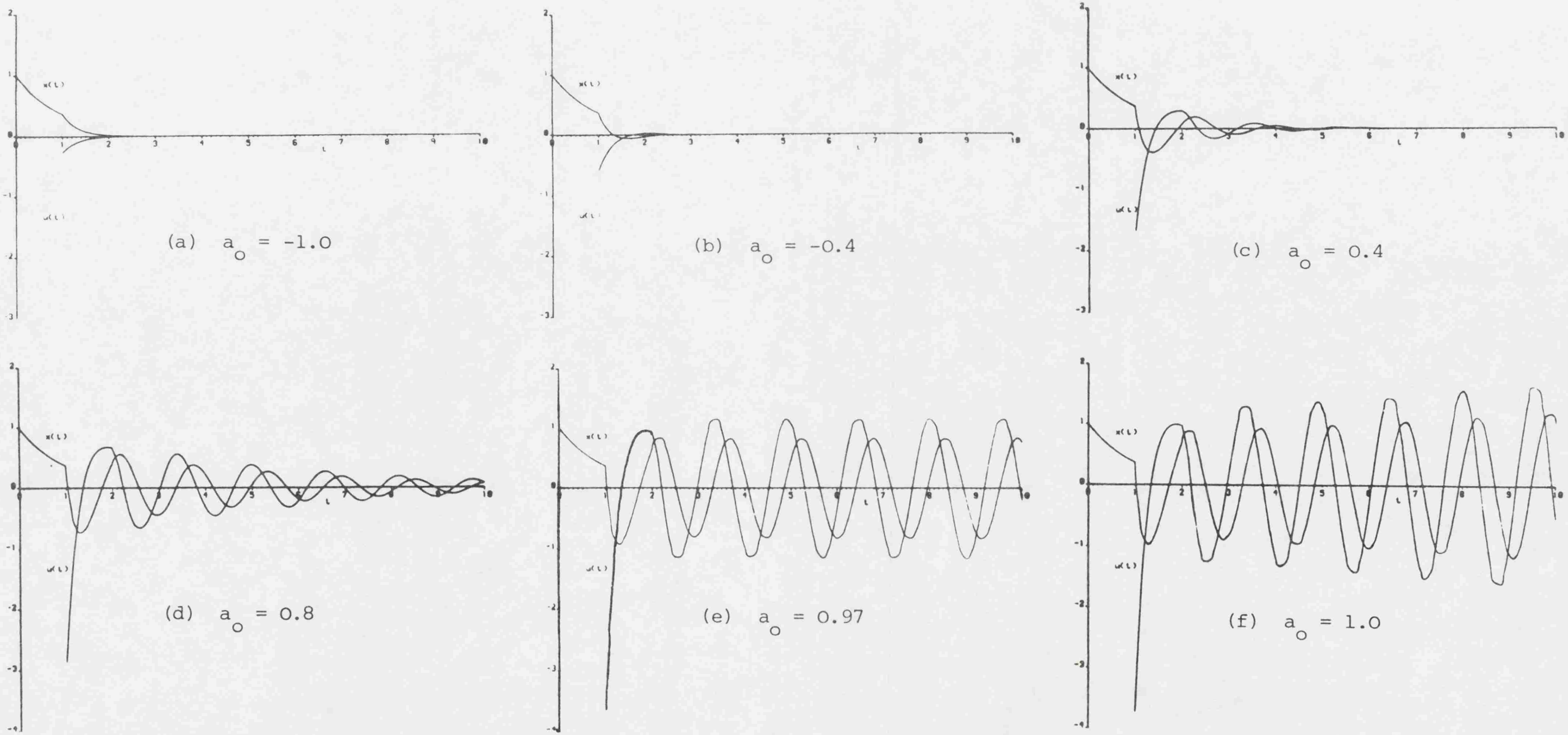
Combining the results of Section 5.1.1, for a stable subplant, Control scheme (1) is stable only if a_o is less than, or equal to, a critical value, a_o^c say. The model values for which Control scheme (1) is stable are referred to as intervals of stability. In this instance, the interval of stability is a semi-infinite interval, with a_o^c, the finite right endpoint, producing a bounded non-convergent output. In physical terms, Control scheme (1) with a stable subplant can be made unstable by incorporating a sufficiently unstable model. Control scheme (1) is now considered when it incorporates an unstable subplant. It is stressed that Control scheme (1) is noise-free, as the presence of noise in a control scheme with an unstable subplant may be particularly detrimental.

5.1.2: An Unstable Subplant

Control scheme (1) is examined when it incorporates plant $(a,b,\tau) = (1,3,1)$. The forms of input (4.25) and output (4.57) produced by underestimates of $a = 1$ are shown in Figure 5.3. As the subplant is unstable its output for the first second, the initial condition response, is strictly increasing. After one second the control becomes available,

FIGURE 5.2: The Input and Output of a <u>*Stable*</u> *Subplant for* <u>*Overestimation*</u> *of a*

$(a,b,\tau) = (-1,3,1), \quad (q,r) = (1,1), \quad x_o = 1$



(a) $a_o = -1.0$

(b) $a_o = -0.4$

(c) $a_o = 0.4$

(d) $a_o = 0.8$

(e) $a_o = 0.97$

(f) $a_o = 1.0$

and in the matched case drives the output to zero along an exponential curve. As a_o is decreased from the matched value the input and output converge less quickly to zero. Eventually, a critical value of a_o is reached, $a_o^{c_1}$ say, for which the input and output converge to non-zero values. In this example $a_o^{c_1} = 0.4311$ and the associated limits of input and output are -0.383 and 1.148 respectively. Any further decrease in a_o below the critical value, results in an unstable control scheme.

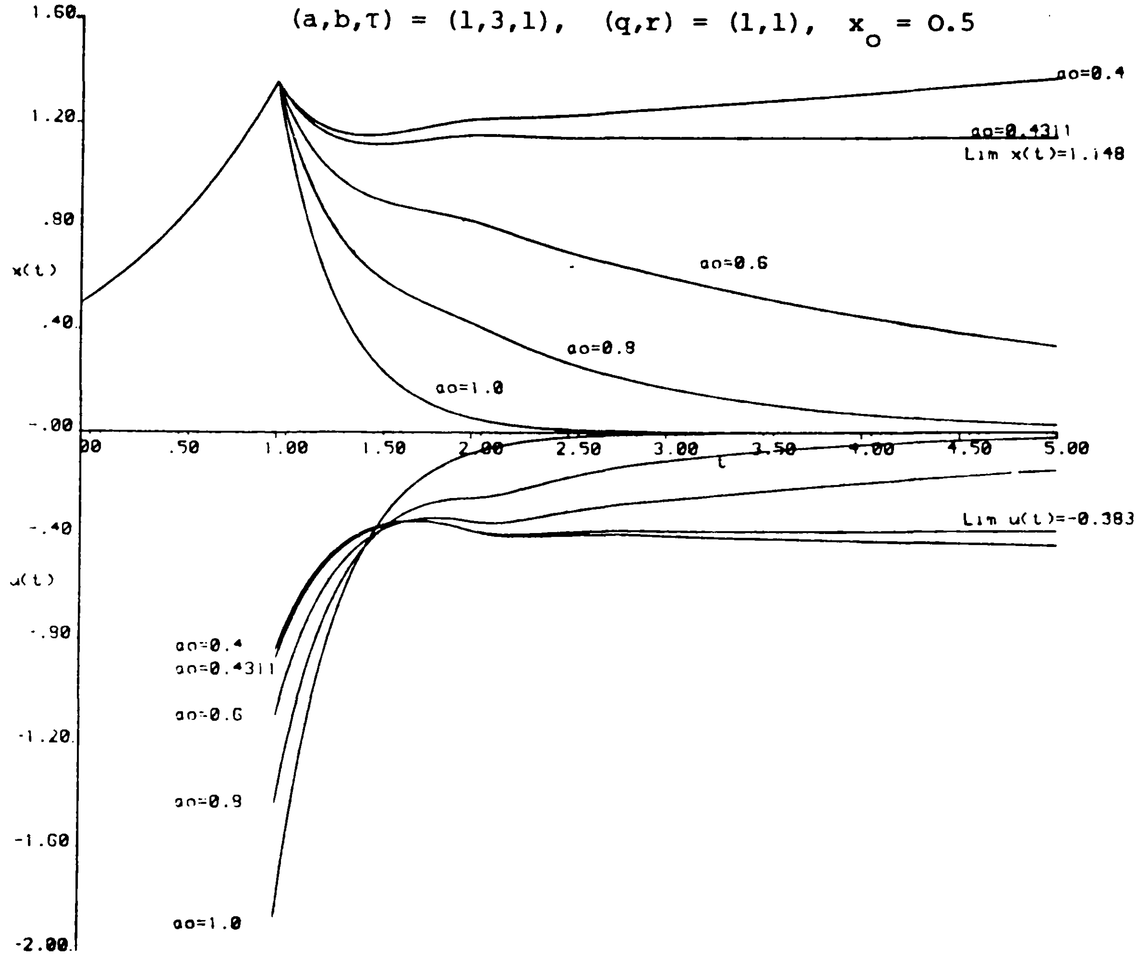


FIGURE 5.3: The Input and Output of an Unstable Subplant for Underestimation of a

The Final Value Theorem (LePage, 1961; Pennisi, 1976) is invoked to determine analytically the critical value $a_o^{c_1}$ and the associated non-zero limits of the input and output. The Final Value Theorem states that if both limits exist

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,\bar{f}(s) \tag{5.1}$$

For mismatch in a, the Laplace transform of the input is given by (4.15) as

$$\bar{u}(s) = \frac{-k_1 x_o (s-a_o) e^{-s\tau}}{(s-c_1)(s-a) - k_1 d_1 e^{-s\tau}} \tag{5.2}$$

and by L'Hôpitals rule (Spivak, 1967)

$$\lim_{s \to 0} s\bar{u}(s) = \begin{cases} 0 & ac_1 - k_1 d_1 \neq 0 \\ \dfrac{-k_1 x_o a_o}{a + c_1 - k_1 d_1 \tau} & ac_1 - k_1 d_1 = 0 \end{cases} \tag{5.3}$$

Assuming the limit exists, the Final Value Theorem gives

$$\lim_{t \to 0} u(t) = \begin{cases} 0 & ac_1 - k_1 d_1 \neq 0 \\ \dfrac{-k_1 x_o a_o}{a + c_1 - k_1 d_1 \tau} & ac_1 - k_1 d_1 = 0 \end{cases} \tag{5.4}$$

Similarly, as (4.56) shows the Laplace transform of the output is given by

$$\bar{x}(s) = \frac{x_o}{s-a} - \frac{b}{s-a} \left\{ \frac{k_1 x_o (s-a_o) e^{-s\tau}}{(s-c_1)(s-a) - k_1 d_1 e^{-s\tau}} \right\} \tag{5.5}$$

using L'Hôpitals rule and the Final Value Theorem, if the limit exists

$$\lim_{t \to \infty} x(t) = \begin{cases} 0 & ac_1 - k_1 d_1 \neq 0 \\ \dfrac{bk_1 x_o a_o}{a(a + c_1 - k_1 d_1 \tau)} & ac_1 - k_1 d_1 = 0 \end{cases} \tag{5.6}$$

The equation $ac_1 - k_1 d_1 = 0$ is satisfied by $a_o^{c_1}$, for which (5.4) and (5.6) take the anticipated values. For $a_o \neq a_o^{c_1}$ the input and output must either have zero limit or no limit which is consistent with Figures 5.1, 5.2 and 5.3.

The effects of overestimating $a = 1$ are catalogued in Figure 5.4. As a_o is increased above the matched value, the input and output become increasingly oscillatory, converging less quickly to zero, see Figures 5.4a,b,c and d. Eventually a critical value of a_o is reached, $a_o^{c_2}$ say, which produces the bounded non-convergent output shown in Figure 5.4e. For the given example the critical value is $a_o^{c_2} = 1.685$ and any increase above this value results in an unstable control scheme, see Figure 5.4f.

Combining the results of Section 5.1.2, for an unstable subplant, Control scheme (1) is stable only for values of a_o on $[a_o^{c_1}, a_o^{c_2}]$, a closed, finite, non-symmetric interval about the matched value. The next section investigates the effects of mismatch in b on the stability of Control scheme (1).

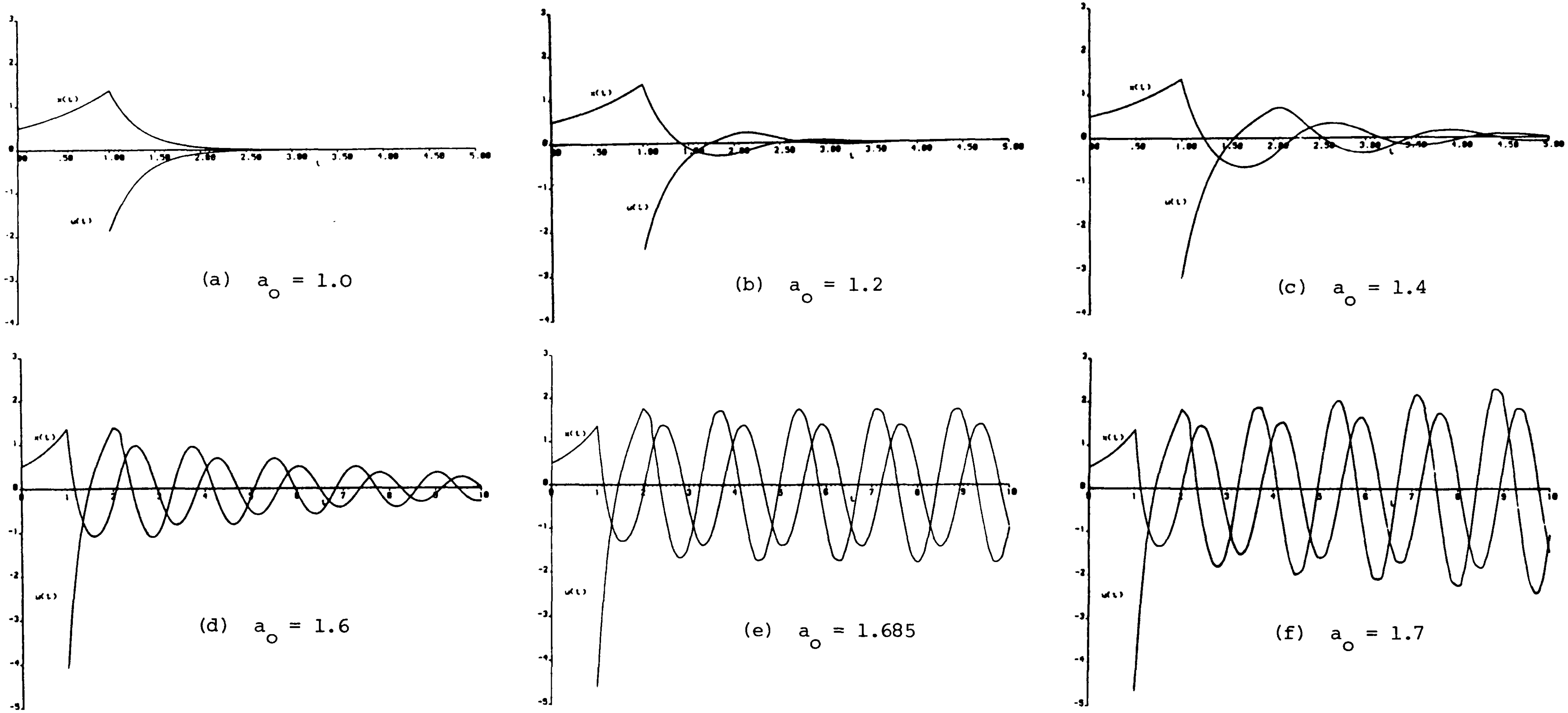
5.2: The Stability of Control Scheme (1) for Mismatch in b

5.2.1: A Stable Subplant

Control scheme (1) is examined when it incorporates plant $(a,b,\tau) = (-1,1,1)$ and the forms of input (4.59) and output (4.60) are shown in Figures 5.5 and 5.6. Figure 5.5 considers positive values of b_o, in which case, the input takes negative values from $t = \tau$. The corresponding output is strictly decreasing, converging to zero more quickly than the initial condition response. Furthermore, for a stable subplant, as b_o approaches zero, input (4.59) tends to the zero function.

FIGURE 5.4: The Input and Output of an <u>Unstable</u> Subplant for <u>Overestimation of a</u>

$(a,b,\tau) = (1,3,1), \quad (q,r) = (1,1), \quad x_o = 0.5$



(a) $a_o = 1.0$ (b) $a_o = 1.2$ (c) $a_o = 1.4$

(d) $a_o = 1.6$ (e) $a_o = 1.685$ (f) $a_o = 1.7$

Figure 5.6 considers negative values of b_o, in which case, the mismatched input takes positive values from $t = \tau$. The corresponding output converges to zero, although less quickly than the initial condition response. The result to be drawn from Figures 5.5 and 5.6 is that for a stable subplant, whatever the value of b_o, Control scheme (1) is stable. The interval of stability is therefore infinite. Attention is now turned to an unstable subplant.

5.2.2: An Unstable Subplant

Control scheme (1) is examined when it incorporates plant $(a,b,\tau) = (1,1,1)$. The forms of input (4.59) and output (4.60) for underestimates of $b = 1$ are shown in Figure 5.7. As b_o is decreased from the matched value the input and output become increasingly oscillatory, converging less quickly to zero, see Figures 5.7a,b,c and d. Eventually a critical value of b_o is reached, $b_o^{c_1}$ say, which produces the bounded, non-convergent output of Figure 5.7e. For the given example, the critical value of b_o is $b_o^{c_1} = 0.713$ and any further decrease results in an unstable control scheme, see Figure 5.7f.

When the subplant is unstable, the effects of overestimating $b = 1$ are shown in Figure 5.8. As b_o is increased from the matched value the input and output converge less quickly to zero. Eventually, a critical value of b_o is attained, $b_o^{c_2}$ say, for which the input and output converge to non-zero values. In this example $b_o^{c_2} = 1.296$ and the associated limits of input and output are -0.2097 and 0.2097 respectively. Any further increase in b_o, above the critical value, produces an unstable control scheme.

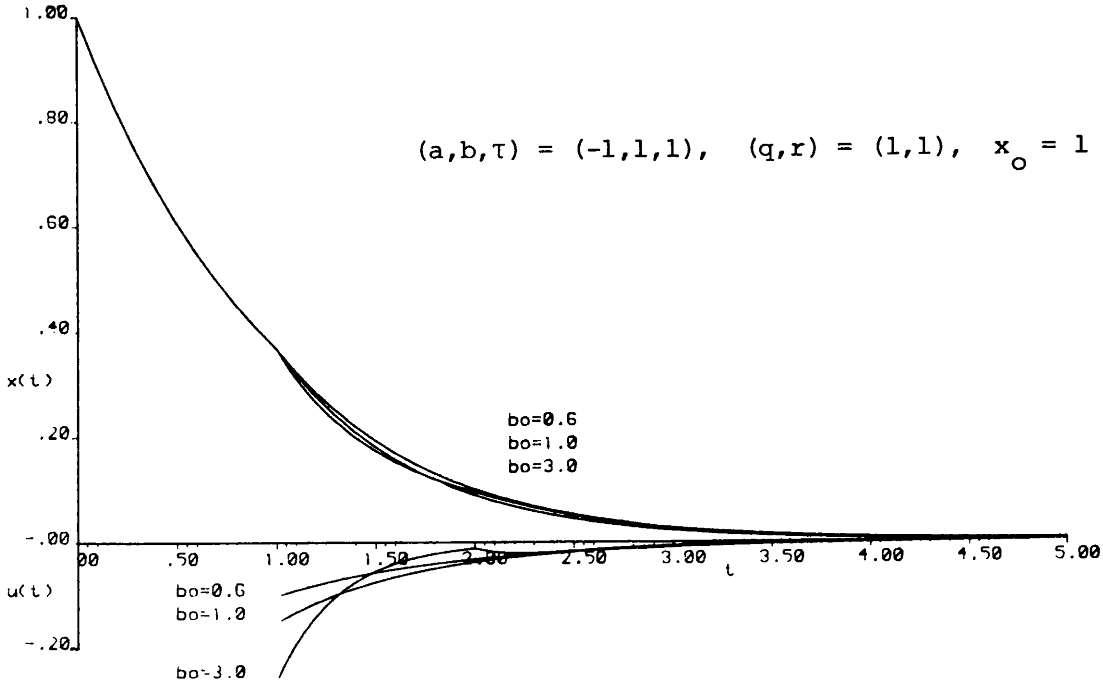


FIGURE 5.5: The Input and Output of a Stable Subplant for Positive values of b_o

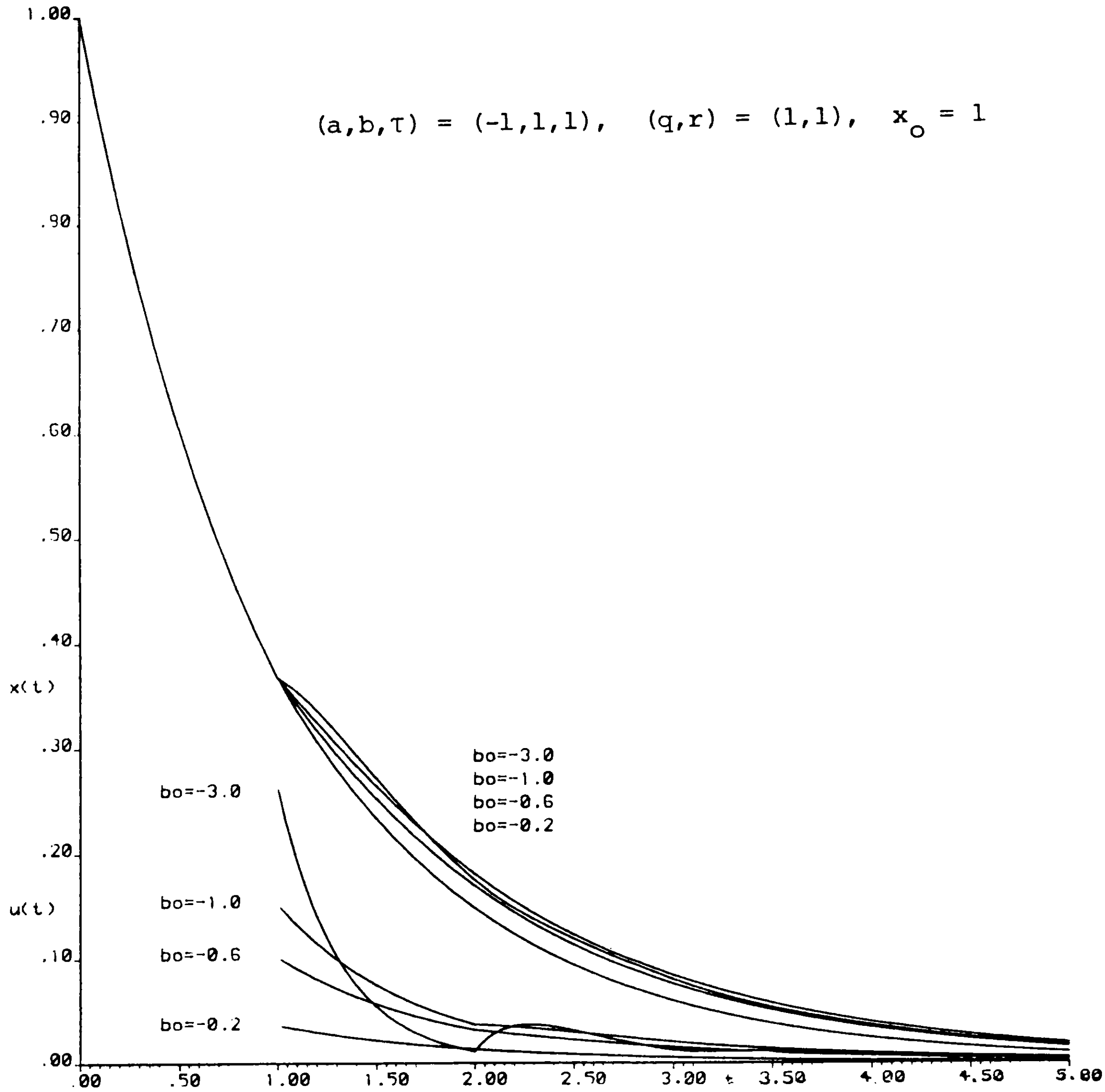
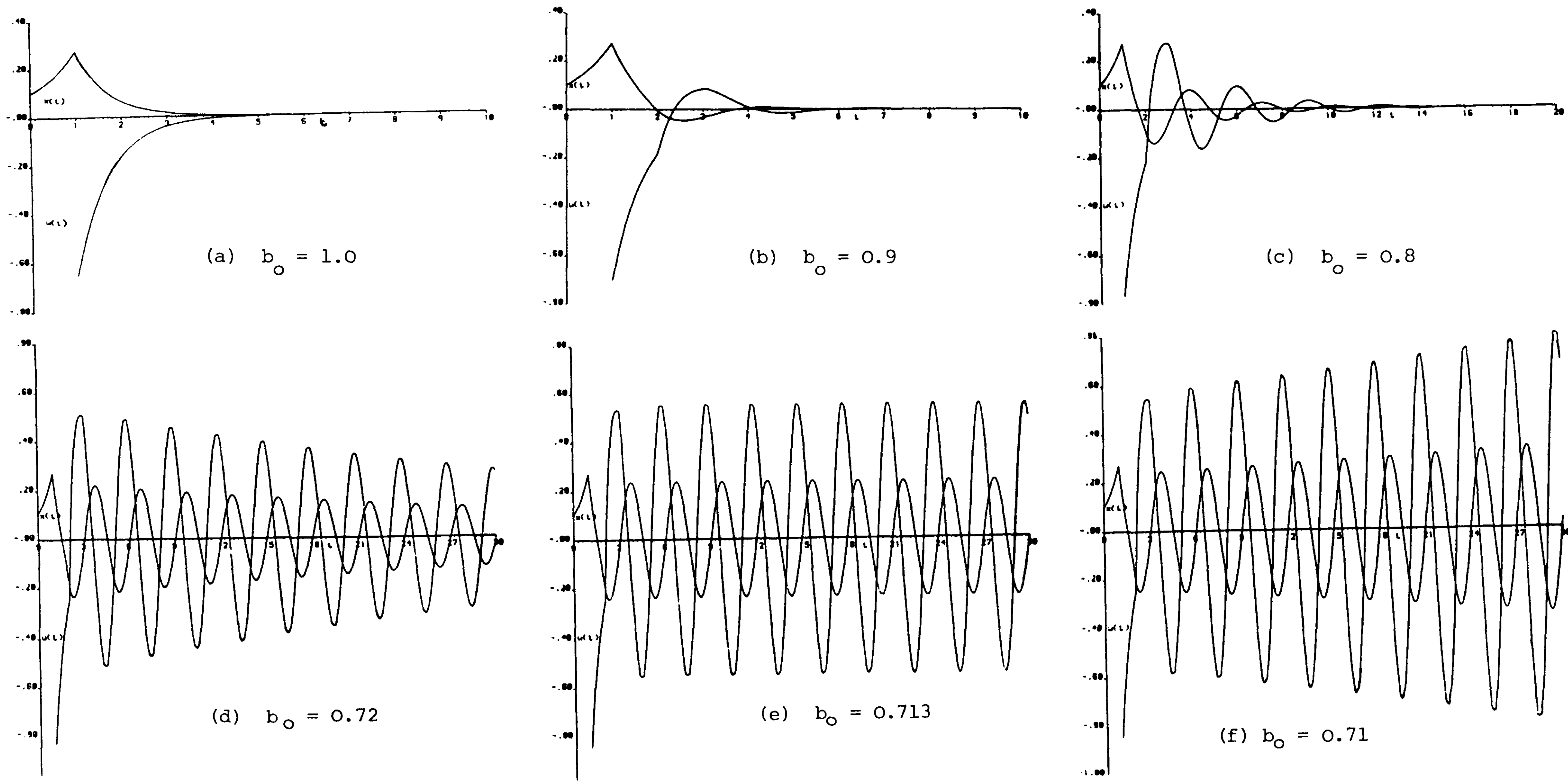


FIGURE 5.6: The Input and Output of a Stable Subplant for Negative values of b_o

FIGURE 5.7: The Input and Output of an Unstable Subplant for Underestimation of b

$(a,b,\tau) = (1,1,1), \quad (q,r) = (1,1), \quad x_o = 0.1$



(a) $b_o = 1.0$

(b) $b_o = 0.9$

(c) $b_o = 0.8$

(d) $b_o = 0.72$

(e) $b_o = 0.713$

(f) $b_o = 0.71$

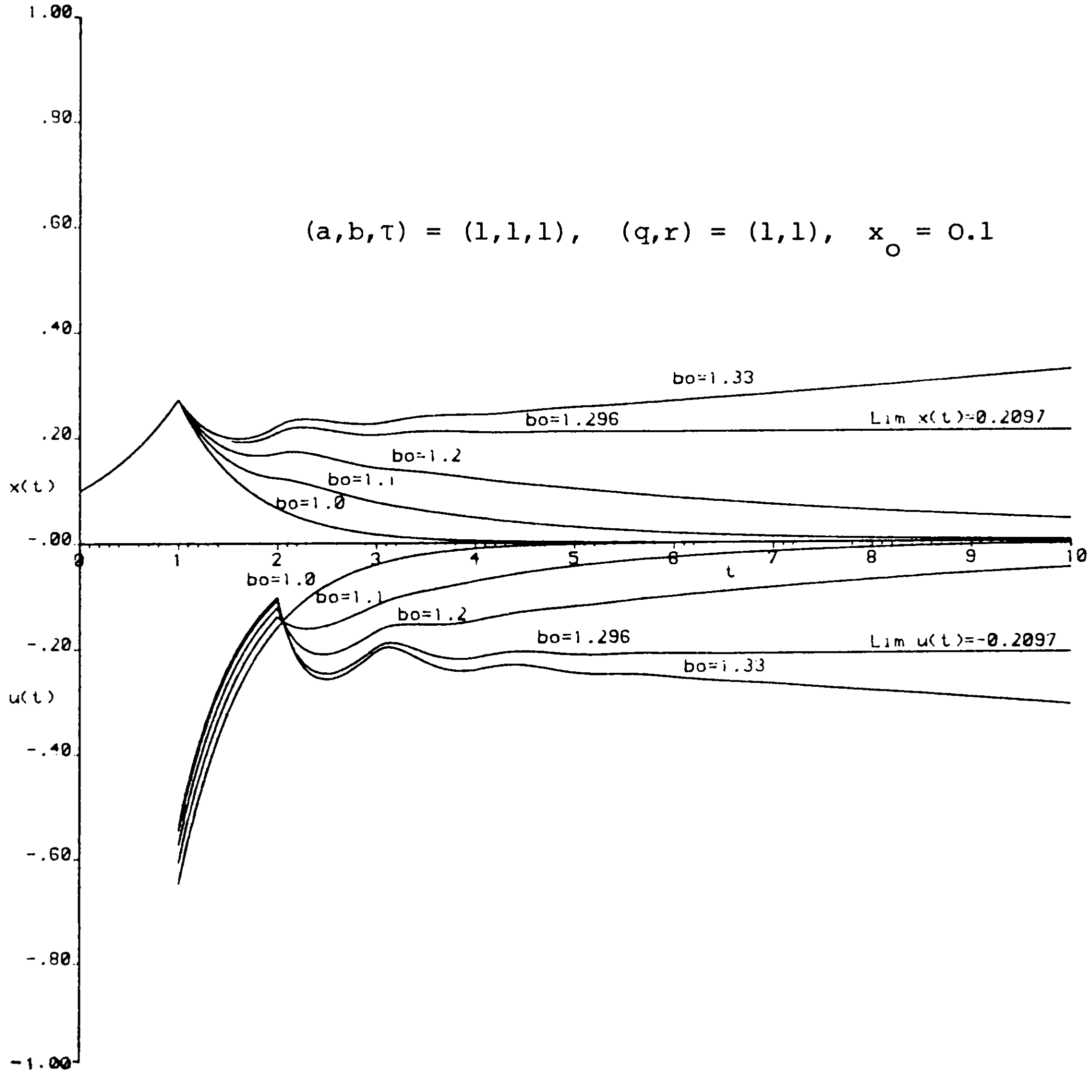


FIGURE 5.8: The Input and Output of an <u>Unstable</u> Subplant for <u>Overestimation of b</u>

As with mismatch in a, the limits of the input and output can be found using (5.1), the Final Value Theorem. If the limits exist, they are given by

$$\lim_{t \to \infty} u(t) = \begin{cases} 0 & c_2 + k_2 d_2 \neq 0 \\ \dfrac{-k_2 x_o}{1 + k_2 d_2 \tau} & c_2 + k_2 d_2 = 0 \end{cases} \tag{5.7}$$

and

$$\lim_{t\to\infty} x(t) = \begin{cases} 0 & c_2+k_2d_2 \neq 0 \\ \dfrac{bk_2x_o}{a(1+k_2d_2\tau)} & c_2+k_2d_2 = 0 \end{cases} \tag{5.8}$$

The equation $c_2+k_2d_2=0$ is satisfied by $b_o^{c_2}$, for which (5.7) and (5.8) take the expected values. For $b_o \neq b_o^{c_2}$, the input and output must either have zero limit or no limit, which is consistent with Figures 5.5, 5.6, 5.7 and 5.8.

Combining the results of Section 5.2.2, with an unstable subplant, Control scheme (1) is stable only for values of b_o on $[b_o^{c_1}, b_o^{c_2}]$, a closed, finite, non-symmetric interval about the matched value. Thus far, mismatch has been in subplant parameters. The next section investigates the effects of mismatch in delay on the stability of Control scheme (1).

5.3: The Stability of Control Scheme (1) for Mismatch in Delay

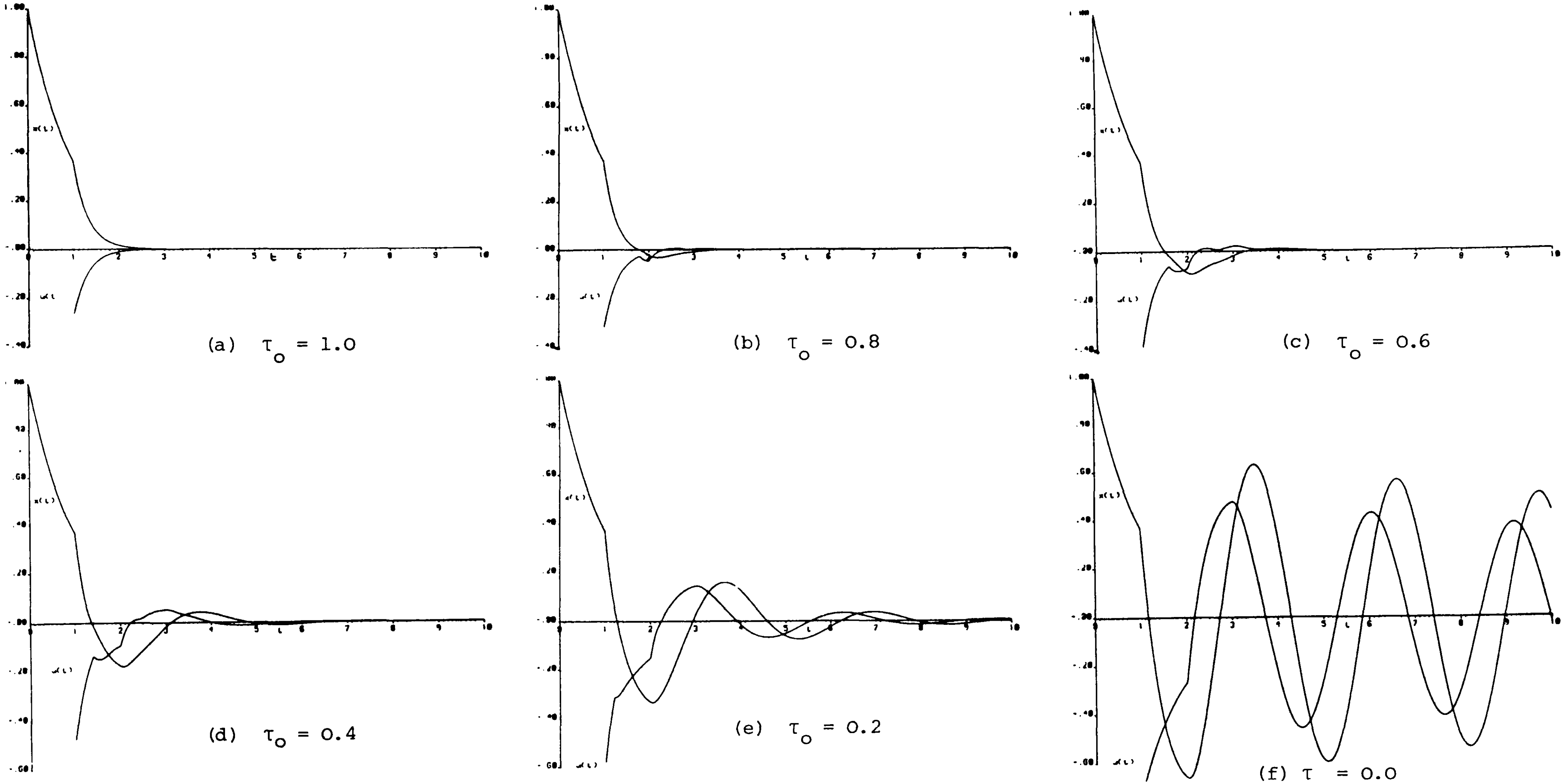
5.3.1: A Stable Subplant

Control scheme (1) is examined when it incorporates plant $(a,b,\tau) = (-1,3,1)$. The forms of input (4.65) and output (4.83) for underestimates of $\tau=1$ are shown in Figure 5.9. Only realisable values of model delay are considered, that is, $\tau_o \geq 0$. As τ_o is reduced from the matched value, the input and output become increasingly oscillatory, converging less quickly to zero, see Figures 5.9a,b,c,d and e. However, even in the extreme case of Figure 5.9f, when the model is delay-free, the input and output converge to zero, ensuring Control scheme (1) remains stable.

When the subplant is stable, the effects of overestimating $\tau=1$

FIGURE 5.9: The Input and Output of a *__Stable__* *Subplant for* *__Underestimation of Delay__*

$(a,b,\tau) = (-1,3,1), \quad (q,r) = (1,1), \quad x_o = 1$



(a) $\tau_o = 1.0$

(b) $\tau_o = 0.8$

(c) $\tau_o = 0.6$

(d) $\tau_o = 0.4$

(e) $\tau_o = 0.2$

(f) $\tau = 0.0$

are shown in Figure 5.10. As the model delay becomes large, input (4.65) tends to the zero function and output (4.83) approaches the initial condition response. Combining the results of Section 5.3.1, with a stable subplant, Control scheme (1) is stable for any realisable model delay. The interval of stability is therefore semi-infinite, with zero as the finite left endpoint.

The results of Section 5.3.1 are consistent with the findings of the authors listed in Sections 1.3 and 1.4, that overestimation of delay is preferable to underestimation, as it produces a more damped response. In particular, recall the paper of Byron, Cox and Ball (1979). Their table of results shows that whilst underestimation of delay increases overshoot, overestimation of delay produces no overshoot, exactly the findings illustrated in Figures 5.9 and 5.10. An unstable subplant is now incorporated into Control scheme (1).

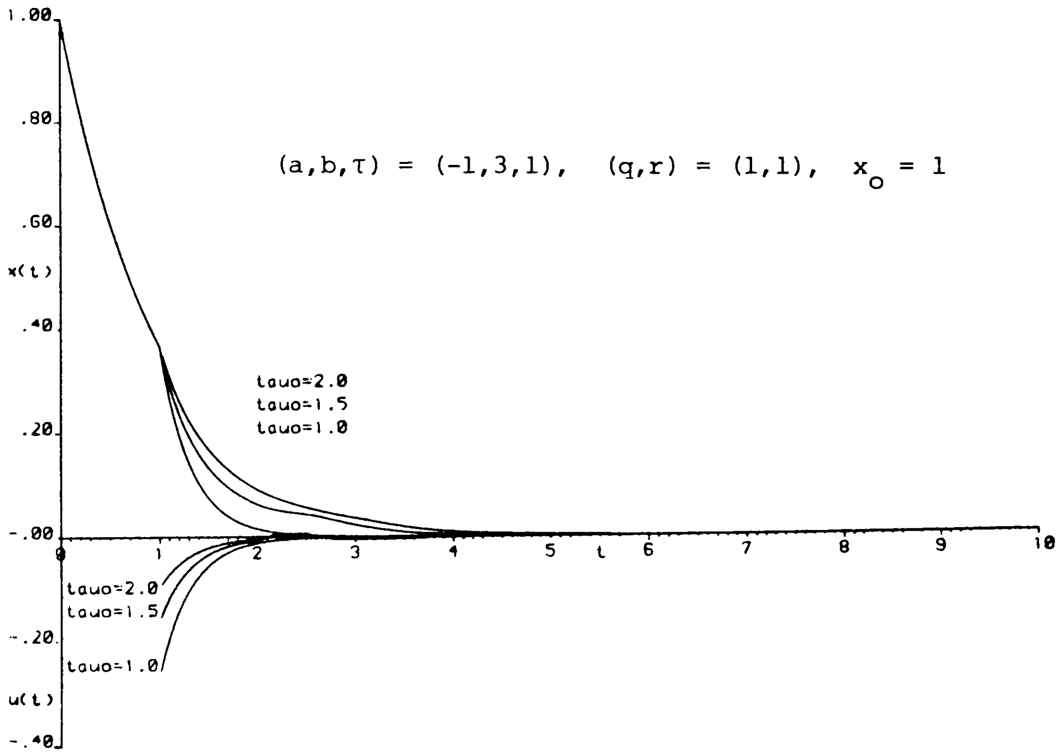


FIGURE 5.10: The Input and Output of a <u>*Stable*</u> *Subplant for* <u>*Overestimation of Delay*</u>

5.3.2: An Unstable Subplant

Control scheme (1) is examined when it incorporates plant (1,3,1). The forms of input (4.65) and output (4.83) for underestimates and overestimates of $\tau = 1$ are shown in Figures 5.11 and 5.12 respectively. When the model delay deviates slightly from the matched value, small spikes appear in the input and output. As mismatch is increased the spikes become larger and more numerous. Eventually, critical values of model delay are attained, $\tau_o^{c_1}$ and $\tau_o^{c_2}$ say, both of which produce bounded, non-convergent outputs. For the given example, Figure 5.11e shows $\tau_o^{c_1} = 0.888$ and Figure 5.12e shows $\tau_o^{c_2} = 1.088$. Any decrease in model delay below $\tau_o^{c_1}$, or increase above $\tau_o^{c_2}$, results in an unstable control scheme, see Figures 5.11f and 5.12f respectively.

Therefore, for an unstable subplant, Control scheme (1) is only stable for values of τ_o on $[\tau_o^{c_1}, \tau_o^{c_2}]$, a closed, finite, non-symmetric interval about the matched value. The fact that $\tau_o^{c_1}$ and $\tau_o^{c_2}$ both produce non-convergent outputs is consistent with the information supplied by (5.1), the Final Value Theorem. For mismatch in delay, the Laplace transform of the output is given by (4.82) as

$$\bar{x}(s) = \frac{x_o}{s-a} - \frac{b}{s-a}\left\{\frac{k_3 x_o e^{-s\tau}}{(s-c) - k_3 b(e^{-s\tau_o} - e^{-s\tau})}\right\} \tag{5.9}$$

in which case

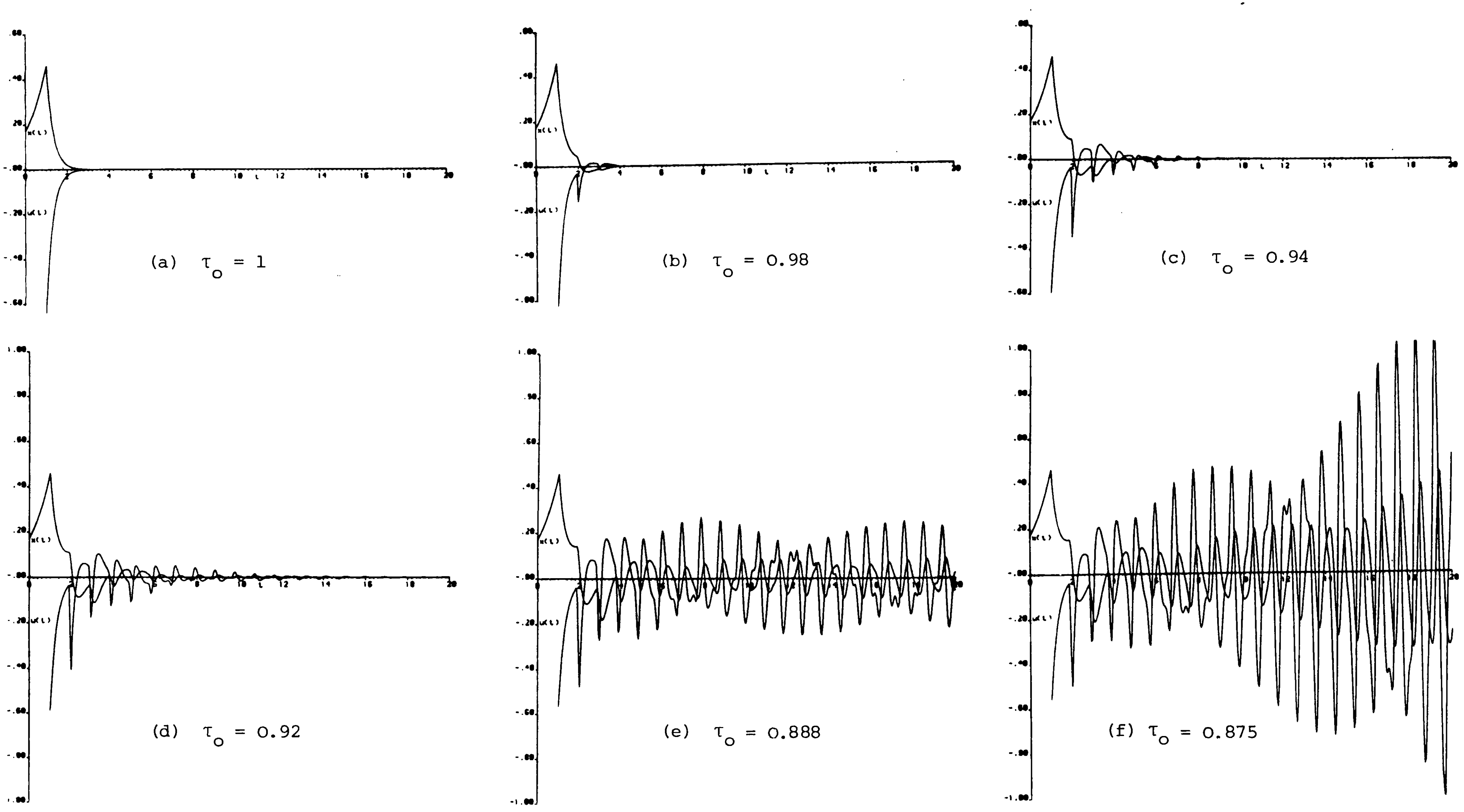
$$\lim_{s \to 0} s\,\bar{x}(s) = 0 \tag{5.10}$$

By the Final Value Theorem, if the limit exists

$$\lim_{t \to \infty} x(t) = 0 \tag{5.11}$$

FIGURE 5.11: The Input and Output of an <u>*Unstable*</u> *Subplant for* <u>*Underestimation of Delay*</u>

$(a,b,\tau) = (1,3,1), \quad (q,r) = (1,1), \quad x_o = 0.17$

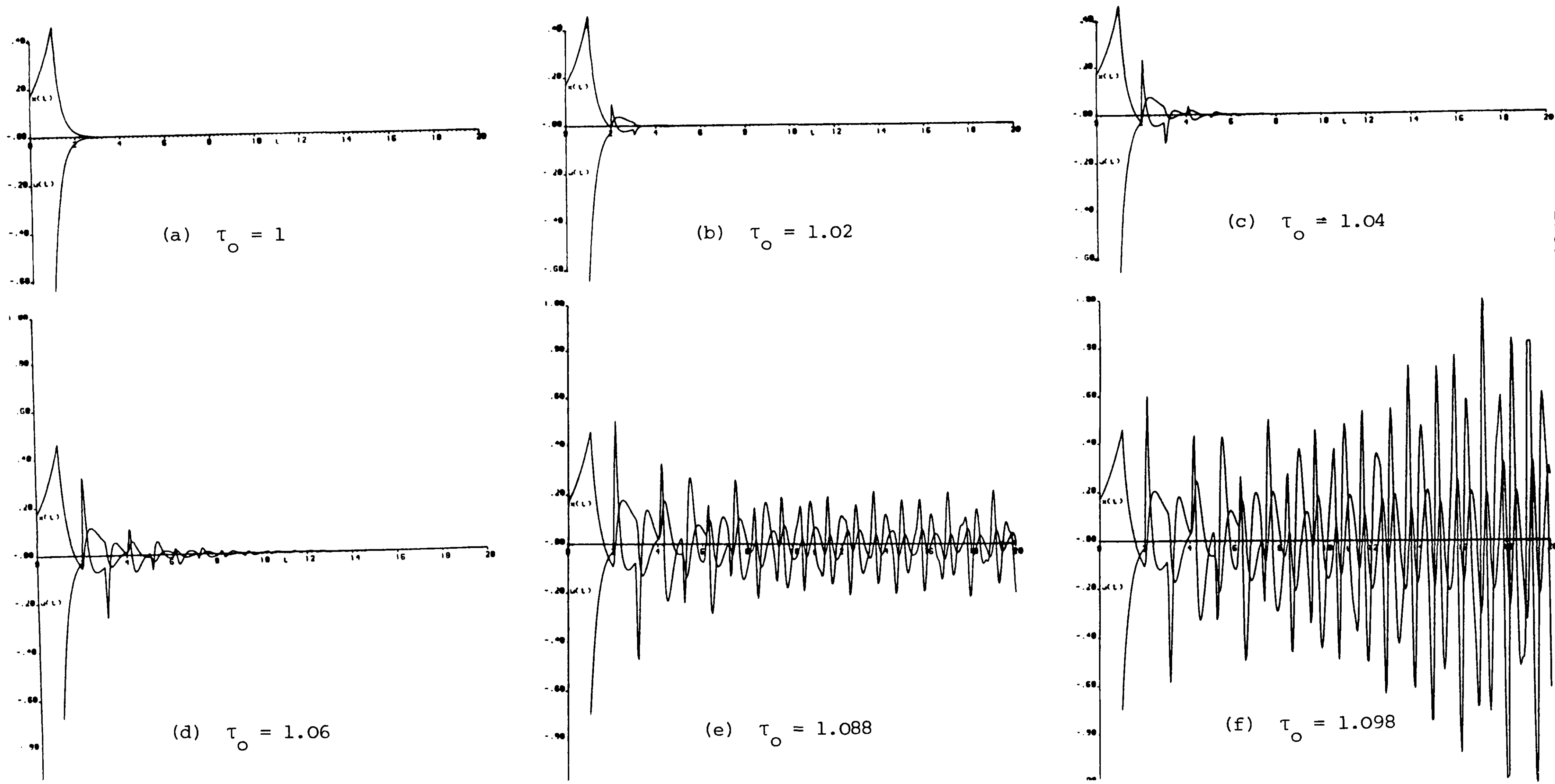


(a) $\tau_o = 1$ (b) $\tau_o = 0.98$ (c) $\tau_o = 0.94$

(d) $\tau_o = 0.92$ (e) $\tau_o = 0.888$ (f) $\tau_o = 0.875$

FIGURE 5.12: The Input and Output for an <u>*Unstable*</u> *Subplant for* <u>*Overestimation of Delay*</u>

$(a,b,\tau) = (1,3,1), \quad (q,r) = (1,1), \quad x_o = 0.17$



which shows zero is the only possible limit for the output, and is consistent with Figures 5.9, 5.10, 5.11 and 5.12. There now follows a short section listing some general results on the intervals of stability.

5.4: General Results on the Intervals of Stability

The different forms of the intervals of stability discovered in Sections 5.1, 5.2 and 5.3 are summarised in Table 5.1. The associated numerical details are given in Table 5.2.

Mismatched Parameter	Subplant Stability	Nature of Intervals of Stability
a	Stable	Semi-infinite interval, finite right endpoint. Right endpoint producing bounded, non-convergent output.
	Unstable	Closed, finite, non-symmetric interval about the matched value. Left endpoint producing a convergent output with a non-zero limit. Right endpoint producing a bounded non-convergent output.
b	Stable	Infinite interval.
	Unstable	Closed, finite, non-symmetric interval about the matched value. Left endpoint producing a bounded, non-convergent output. Right endpoint producing a convergent output with a non-zero limit.
τ	Stable	Semi-infinite interval with zero as the finite left endpoint.
	Unstable	Closed, finite, non-symmetric interval about the matched value. Both endpoints producing bounded non-convergent outputs.

TABLE 5.1: Nature of Intervals of Stability

Plant (a,b,τ)	Mismatched Parameter	Interval of Stability
(-1,3,1)	$a = -1$	$(-\infty, 0.970]$
(1,3,1)	$a = 1$	$[0.431, 1.685]$
(-1,1,1)	$b = 1$	$(-\infty, \infty)$
(1,1,1)	$b = 1$	$[0.713, 1.296]$
(-1,3,1)	$\tau = 1$	$[0, \infty)$
(1,3,1)	$\tau = 1$	$[0.888, 1.088]$

TABLE 5.2: Numerical Details

An investigation is now undertaken into how the length of an interval of stability is affected by changes in subplant parameters. The interval of stability considered is $[b_o^{c_1}, b_o^{c_2}]$ of Section 5.2.2, which comprises values of b_o for which Control scheme (1) is stable when it incorporates an unstable subplant. From Table 5.2 it is seen that for plant $(a,b,\tau) = (1,1,1)$ the interval of stability is $[0.713, 1.296]$. Any increase in parameter a makes the subplant more unstable, and results in a decrease in the length of the interval of stability. This is shown when Control scheme (1) incorporates plants $(a,b,\tau) = (2,1,1)$ and $(a,b,\tau) = (3,1,1)$ for which the intervals of stability are $[0.906, 1.090]$ and $[0.967, 1.030]$ respectively. For the plant $(a,b,\tau) = (3,1,1)$ mismatching b by as little as 5% will produce an unstable control scheme.

If the interval of stability approximates to values of b_o within a certain percentage of b, for an increase in b an increase in the interval of stability is anticipated. For the original plant $(a,b,\tau) = (1,1,1)$ the interval of stability is $[0.713, 1.296]$. When Control scheme (1) incorporates plants $(a,b,\tau) = (1,2,1)$ and $(a,b,\tau) = (1,3,1)$ the corresponding intervals of stability increase to

[1.450, 2.759] and [2.200, 4.285] as expected. These results are collected together in Table 5.3. The next section examines the relationship between mismatch and the performance of Control scheme (1).

Parameter Varied	Plant	Interval of Stability for Mismatch in b
a	(1,1,1) (2,1,1) (3,1,1)	[0.713, 1.296] [0.906, 1.090] [0.967, 1.030]
b	(1,1,1) (1,2,1) (1,3,1)	[0.713, 1.296] [1.450, 2.759] [2.200, 4.285]

TABLE 5.3: Variation of Intervals of Stability with Subplant Parameters

5.5: The Performance of Control Scheme (1) in the presence of Mismatch

The numerically calculated graphs of cost functional against model value illustrate the relationship between performance and mismatch, see Figures 5.13 and 5.14. For each case of mismatch, a global minimum is produced when the model parameter takes the matched value, as, in this instance, the output is the solution of an LQP problem. The curves are similar to those of Chotai (1980, 1981) produced by analogue simulation.

When Control scheme (1) incorporates a stable subplant, mismatch has the following effect on the quadratic cost functional. Figure 5.13a shows that overestimating parameter a produces the more dramatic increase in cost. This may correspond to using an unstable model for a stable subplant, and is consistent with the fact that such an arrangement can produce an unstable control scheme. Figure 5.14a shows that underestimating the time delay produces the greater increase in cost. This is consistent with the input and output becoming more oscillatory as the model delay approaches zero. As a result of mismatch in b, variations in the cost functional are small, which is consistent with Control scheme (1) incorporating a stable subplant remaining stable for all values of model parameter b_o.

When Control scheme (1) incorporates an unstable subplant, as mismatch is increased the cost diverges to infinity, see Figures 5.13b and 5.14b. The high cost is incurred by inputs and outputs with large deviations from the exponentially decaying matched functions.

It is observed from Figures 5.13a and 5.14a that when Control scheme (1) incorporates a stable subplant the graphs of the cost functional tend to limiting values as parameter a_o becomes large and negative and as the model delay becomes large and positive. These limiting values may be calculated analytically as they correspond to the case where the input is the zero function. In this instance the output is the initial condition response and the cost functional

$$J(u(t)) = \int_0^\infty \{qx^2(\sigma) + ru^2(\sigma)\}\, d\sigma \tag{5.12}$$

reduces to

$$\begin{aligned} J(0) &= \int_0^\infty qe^{2a\sigma}\, x_o^2\, d\sigma \\ &= \frac{qx_o^2}{2(-a)}, \quad a < 0 \end{aligned} \tag{5.13}$$

Therefore, with the choice of parameters adopted for stable subplants throughout this chapter

$$\lim_{a_o \to -\infty} J(u) = \lim_{\tau_o \to \infty} J(u) = \frac{1}{2(-a)}, \quad a < 0 \tag{5.14}$$

The values of (5.14) for the examples of mismatch in a and mismatch in delay are given in Tables 5.4 and 5.5 respectively, and correspond well with the numerically calculated graphs.

a	$\lim_{a_o \to -\infty} J(u)$
-1	0.5
-0.5	1.0
-0.3	1.66
-0.2	2.5

TABLE 5.4: Limiting Values of Cost Functional for Mismatch in a

a	$\lim_{\tau_o \to \infty} J(u)$
-1	0.5
-0.95	0.5263
-0.9	0.555

TABLE 5.5: Limiting Values of Cost Functional for Mismatch in Delay

An interpretation of the existence of these limits is that for a stable subplant, if mismatch takes the form of an overestimate of the plant delay, or an underestimate of parameter a, it is always better to implement Control scheme (1) rather than have no control action. Conversely, Figures 5.13a and 5.14a show that by either underestimating the plant delay, or overestimating parameter a, Control scheme (1) may incur a cost which is greater than the limiting value associated with the zero input. In this instance, the policy of no control action is preferable to implementing the input generated by Control scheme (1).

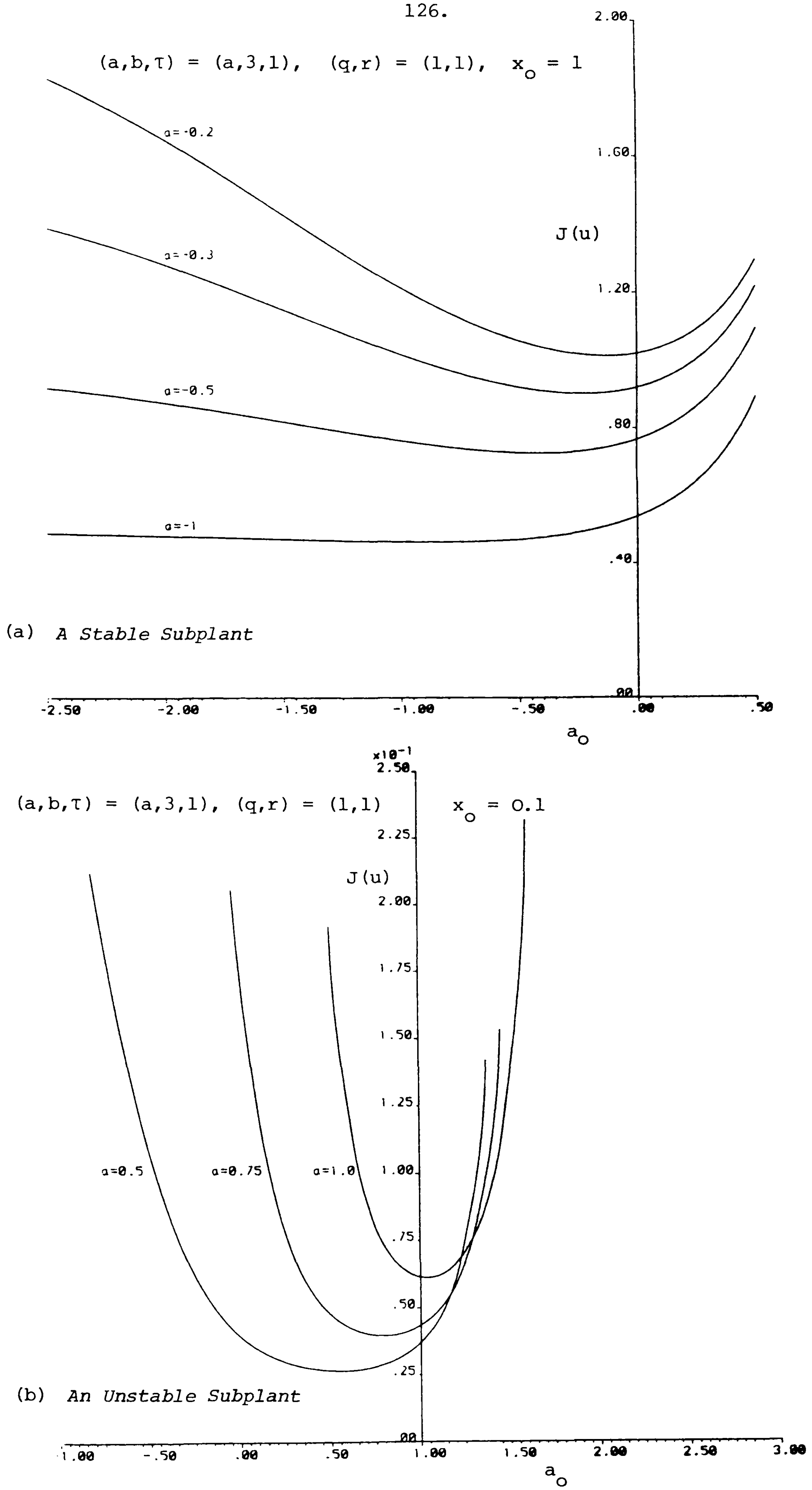


FIGURE 5.13: Variation in Cost with Mismatch in a

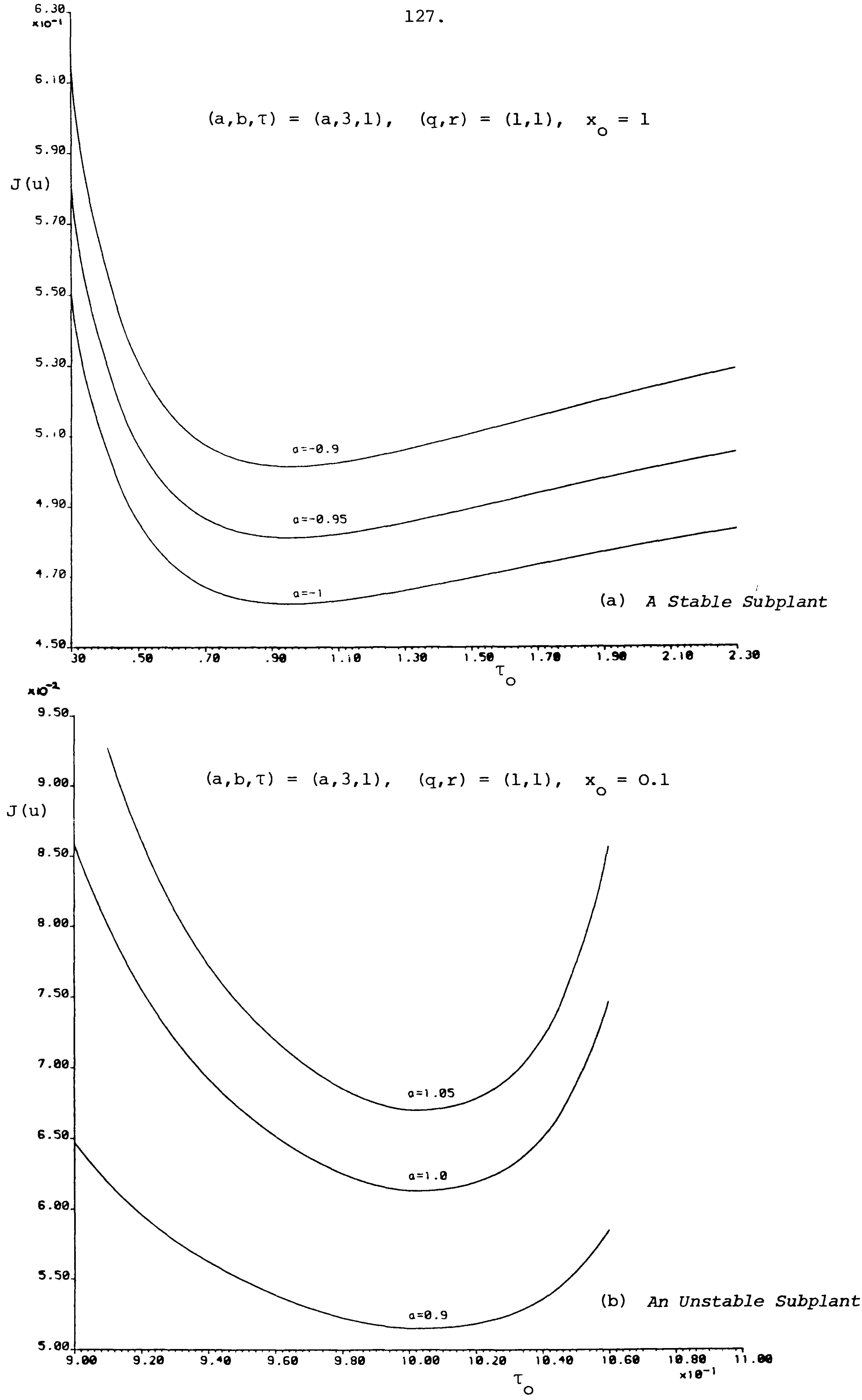


FIGURE 5.14: Variation in Cost with Mismatch in Delay

Conclusions

From Chapter 3, Control scheme (1) is defined as stable if it produces a bounded output, and matched Control scheme (1) is stable irrespective of the stability of the subplant. The results of Chapter 5 are obtained for first-order subplants, from the expressions for input and output in Chapter 4. When Control scheme (1) incorporates a stable subplant it is stable for any value of mismatch in b and any realisable value of mismatch in delay. However, Control scheme (1) with a stable subplant can be made unstable when mismatch in parameter a creates a sufficiently unstable model. When Control scheme (1) incorporates an unstable subplant, it is only stable for a closed, finite, non-symmetric interval of model parameter values about the matched value. It is observed that increasing the instability of the subplant reduces the length of the interval of stability.

All the curves representing the variation of cost with model parameter have a minimum at the matched value. When Control scheme (1) incorporates a stable subplant, if the matched values are not available, the best strategy is to underestimate parameter a and overestimate the delay. When Control scheme (1) incorporates an unstable subplant, as mismatch increases the cost diverges to infinity.

Chapter 6: The Optimal Control of Linear Systems with Control and Measurement Delays

Chapter 6: The Optimal Control of Linear Systems with Control and Measurement Delays

Introduction

Chapter 6 considers the optimisation of a quadratic cost functional when the associated plant contains delays in both control and measurement. Section 6.1 presents predictor Control scheme (3) which is optimal when matched, but requires prior knowledge of the initial state of the subplant. Section 6.2 presents an alternative predictor control scheme, Control scheme (4), which is a natural extension of Control scheme (2). Although matched Control scheme (4) is suboptimal, it does not require prior knowledge of the initial state of the subplant.

Section 6.3 is a comparison of the performances of matched Control schemes (3) and (4) for variations in the measurement delay. The performance improvement achieved by matched Control scheme (3) may therefore be weighed against the difficulties of obtaining and implementing prior knowledge of the initial state of the subplant. As matched Control scheme (4) is suboptimal, mismatch may improve performance (Vit, 1979; Ioannides, Rogers and Latham, 1979; Hocken, Salehi and Marshall, 1983). Section 6.4 considers the types of mismatch which may produce this improvement. The contents of Chapter 6 appear in shortened form in Hocken and Marshall (1982).

6.1: An Optimal Control Scheme

As in Chapter 3, the plant to be controlled incorporates noise-free, time-varying, linear subplant (3.1)

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_o) = x_o \tag{6.1}$$

defined on a fixed, finite time interval $[t_o, t_f]$. The symbol $A(t)$ is an $n \times n$ matrix, $B(t)$ is an $n \times m$ matrix, $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$. The subplant together with delays τ_c in control and τ_m in measurement form the plant, see Figure 6.1.

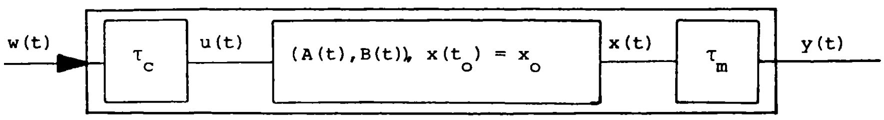


FIGURE 6.1: The Plant, incorporating Subplant and Delays

As in earlier work, it is assumed the initial function stored in the delays is the zero function and the connections between the subplant and the delays are inaccessible. Again it is required to design a control scheme which minimises quadratic cost functional (3.2)

$$J(u) = \langle x(t_f),\, Gx(t_f)\rangle + \int_{t_o}^{t_f} \{\langle x(\sigma),\, Q(\sigma)x(\sigma)\rangle + \langle u(\sigma),\, R(\sigma)u(\sigma)\rangle\}d\sigma \tag{6.2}$$

where $G, Q(t) \geqslant 0$ and $R(t) > 0$ are symmetric matrices and t_f is the fixed final time.

As the cost functional, subplant and control delay are uncnanged from the time-varying case of Chapter 3, the optimal subplant input is given by (3.38); the solution to a LQP problem with initial conditions

$$x(t_o+\tau_c) = \Phi(t_o+\tau_c, t_o)x_o \tag{6.3}$$

The measurement delay serves only to complicate the construction of optimal subplant input

$$u(t) = -L(t)x(t) \qquad t_o+\tau_c \le t \le t_f \tag{6.4a}$$

where

$$L(t) = R^{-1}(t)B^T(t)K(t) \qquad t_o+\tau_c \le t \le t_f \tag{6.4b}$$

A predictor control scheme provides the advanced version of the state necessary for the desired plant input. However, to overcome the effects of the measurement delay, it is necessary to choose the initial state of the model to be an advanced version of the initial state of the subplant. This requires prior knowledge of the initial state of the subplant. Matched Control scheme (3) of Figure 6.2 utilises this prior knowledge, together with a switch in the outer feedback loop, to produce optimal subplant input (6.4). Mismatch is not introduced in this chapter until Section 6.4. Therefore, in Sections 6.1, 6.2 and 6.3, the prefix 'matched' will be dropped from 'the matched control scheme'.

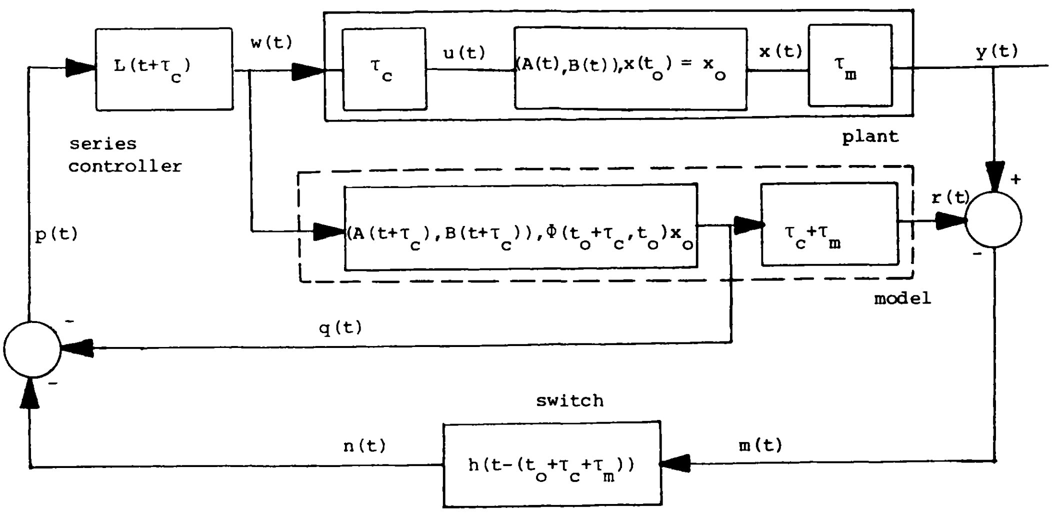


FIGURE 6.2: Control Scheme (3)

Predictor Control scheme (3) incorporates a time-varying model with an accessible connection between model subplant and model delay. The model subplant is an advanced version of the actual subplant, and has an initial state

$$q(t_o) = \Phi(t_o + \tau_c, t_o)x_o \tag{6.5}$$

The model subplant and the series controller may be defined arbitrarily for $t \in (t_f - \tau_c, t_f]$ as they do not affect the cost functional on this interval. The following is a mathematical analysis of Control scheme (3) showing how it achieves optimal subplant input (6.4).

The subplant output is that of a time-varying linear system

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma \tag{6.6}$$

and the zero function in the measurement delay ensures the plant output is given by

$$y(t) = x(t-\tau_m)h(t-(t_o+\tau_m)) \tag{6.7}$$

The model contains an advanced version of the subplant, with an advanced initial state, in which case its delay-free output is given by

$$q(t) = \Phi(t+\tau_c,t_o+\tau_c)q(t_o) + \int_{t_o}^{t} \Phi(t+\tau_c,\sigma+\tau_c)B(\sigma+\tau_c)w(\sigma)d\sigma \tag{6.8}$$

As the model input is an advanced version of the subplant input, (6.8) may be expressed as

$$q(t) = \Phi(t+\tau_c,t_o)x_o + \int_{t_o}^{t} \Phi(t+\tau_c,\sigma+\tau_c)B(\sigma+\tau_c)u(\sigma+\tau_c)d\sigma \tag{6.9}$$

and with a change of variable (6.9) may be written as

$$q(t) = \Phi(t+\tau_c, t_o)x_o + \int_{t_o+\tau_c}^{t+\tau_c} \Phi(t+\tau_c, \sigma)B(\sigma)u(\sigma)d\sigma \tag{6.10}$$

Furthermore, as the subplant input is zero on the interval $[t_o, t_o+\tau_c)$ the lower limit of integration may be taken as t_o, in which case

$$q(t) = x(t+\tau_c) \tag{6.11}$$

an advanced version of subplant output (6.6).

From (6.11), the delayed output of the model is

$$r(t) = x(t-\tau_m)h(t-(t_o+\tau_c+\tau_m)) \tag{6.12}$$

and subtracting (6.12) from (6.7) shows the term entering the outer feedback loop is given by

$$m(t) = x(t-\tau_m)\{h(t-(t_o+\tau_m))-h(t-(t_o+\tau_c+\tau_m))\} \tag{6.13}$$

The term (6.13) is only non-zero for $t_o+\tau_m \le t < t_o + \tau_c + \tau_m$, and as the switch in the outer feedback loop is open for $t_o \le t < t_o + \tau_c + \tau_m$, the term leaving the outer feedback loop is the zero function

$$n(t) = m(t)h(t-(t_o+\tau_c+\tau_m)) = 0 \tag{6.14}$$

The term operated on by the series controller is therefore given by

$$p(t) = -q(t) = -x(t+\tau_c) \tag{6.15}$$

in which case

$$w(t) = -L(t+\tau_c)x(t+\tau_c) \tag{6.16}$$

and the zero function in the control delay yields the optimal subplant input

$$u(t) = -L(t)x(t)h(t-(t_o+\tau_c)) \tag{6.17}$$

Repeating (3.51), (3.52) and (3.53) for completeness; implementing optimal control (6.17), subplant equation (6.1) takes the form

$$\dot{x}(t) = \{A(t)-B(t)L(t)h(t-(t_o+\tau_c))\}x(t), \quad x(t_o)=x_o \tag{6.18}$$

the solution of which is

$$x(t) = \begin{cases} \Phi(t,t_o)x_o & t_o \le t < t_o + \tau_c \\ \psi(t,t_o+\tau_c)\Phi(t_o+\tau_c,t_o)x_o & t_o+\tau_c \le t \le t_f \end{cases} \tag{6.19}$$

where $\psi(t,\sigma)$ is the transition matrix associated with $(A-BL)(t)$. From (6.17), the corresponding optimal subplant input is given by

$$u(t) = \begin{cases} 0 & t_o \le t < t_o+\tau_c \\ -L(t)\psi(t,t_o+\tau_c)\Phi(t_o+\tau_c,t_o)x_o & t_o+\tau_c \le t \le t_f \end{cases} \tag{6.20}$$

Although the outer feedback loop contains the zero function, it is useful in that it will contain mismatch information should the plant not be modelled exactly. It has been stressed that Control scheme (3) requires prior knowledge of the initial state of the subplant, so it can be incorporated into the plant model. Section 6.2 presents an alternative predictor control scheme, Control scheme (4), which is a natural extension of Control scheme (2). Although Control scheme (4) is suboptimal, it does not require prior knowledge of the initial state of the subplant.

6.2: An Alternative Control Scheme

In the absence of prior knowledge of the initial state of the subplant, to produce a subplant input of feedback form it is necessary to await the initial condition response of the subplant. Therefore, the time before a feedback subplant input may be implemented is the sum of the delay in control and the delay in measurement. Consequently, the optimal subplant input (6.20) is not available in this situation. Restricting attention to subplant inputs implemented after time $t = t_o + \tau_c + \tau_m$, the desired subplant input is the solution of a LQP problem with initial conditions

$$x(t_o+\tau_c+\tau_m) = \Phi(t_o+\tau_c+\tau_m, t_o)x_o \tag{6.21}$$

Section 3.1 shows the Riccati equations (3.4) are independent of any initial conditions, in which case the desired subplant input is given by

$$u(t) = -L(t)x(t) \qquad t_o + \tau_c + \tau_m \leqslant t \leqslant t_f \tag{6.22}$$

Control scheme (4) of Figure 6.3 is a natural extension of Control scheme (2) and realises desired input (6.22).

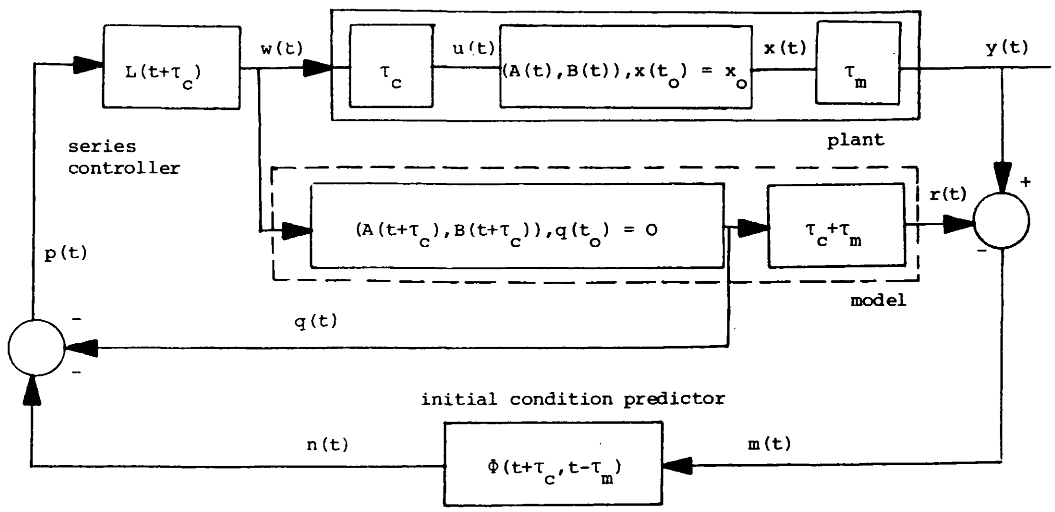


FIGURE 6.3: Control scheme (4)

In the absence of prior knowledge of the initial state of the subplant, the initial state of the model is fixed as zero. Again the model subplant, initial condition predictor and series controller may be defined arbitrarily for $t \in (t_f - \tau_c, t_f]$ as they do not affect the cost functional on this interval. Although the model subplant and series controller are defined for $t \in [t_o, t_o + \tau_m)$ the measurement delay ensures they have zero inputs on this interval. Therefore, cost functional (6.2) is minimised over $[t_o + \tau_c + \tau_m, t_f]$ by Control scheme (4) which effectively operates on $[t_o + \tau_m, t_f - \tau_c]$. Furthermore, the initial condition predictor is defined for $t \in [t_o + \tau_m, t_f - \tau_c]$. A mathematical analysis of Control scheme (4) is now provided to show how it produces the desired subplant input (6.22).

As there is no reference input, the plant input is generated solely by feeding back the plant output. For the interval $[t_o, t_o+\tau_m)$, the zero function in the measurement delay ensures there is zero plant output and consequently zero plant input

$$w(t) = 0 \qquad t_o \leq t < t_o + \tau_m \tag{6.23}$$

After time $t = t_o+\tau_m$ the appearance of the subplant initial condition response as plant output generates a plant input. The zero function in the control delay ensures there is no subplant input for a further τ_c seconds

$$u(t) = 0 \qquad t_o \leq t < t_o + \tau_c + \tau_m \tag{6.24}$$

Therefore, any subsequent integral expression containing $w(\sigma)$ or $u(\sigma)$ as a factor of the integrand is zero for σ on the intervals $[t_o, t_o+\tau_m)$ and $[t_o, t_o+\tau_c+\tau_m)$ respectively.

The subplant output is that of a time-varying linear system

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma \tag{6.25}$$

and the zero function in the measurement delay ensures the plant output is given by

$$y(t) = x(t-\tau_m)h(t-(t_o+\tau_m)) \tag{6.26}$$

The model contains an advanced version of the subplant, with zero initial state, in which case its delay-free output is given by

$$q(t) = \int_{t_o}^{t} \Phi(t+\tau_c,\sigma+\tau_c)B(\sigma+\tau_c)w(\sigma)d\sigma$$

$$= \int_{t_o}^{t} \Phi(t+\tau_c,\sigma+\tau_c)B(\sigma+\tau_c)u(\sigma+\tau_c)d\sigma$$

$$= \int_{t_o+\tau_c}^{t+\tau_c} \Phi(t+\tau_c,\sigma)B(\sigma)u(\sigma)d\sigma \tag{6.27}$$

As the subplant input is zero on the interval $[t_o,t_o+\tau_c+\tau_m)$, the lower limit of integration may be taken as t_o

$$q(t) = \int_{t_o}^{t+\tau_c} \Phi(t+\tau_c,\sigma)B(\sigma)u(\sigma)d\sigma \tag{6.28}$$

which shows the delay-free model output is an advanced version of the forced part of the subplant output (6.25).

From (6.28), the delayed model output

$$r(t) = \int_{t_o}^{t-\tau_m} \Phi(t-\tau_m,\sigma)B(\sigma)u(\sigma)d\sigma \tag{6.29}$$

is the forced part of plant output (6.26). Subtracting (6.29) from (6.26) shows the term entering the outer feedback loop is a delayed version of the subplant initial condition response

$$m(t) = \Phi(t-\tau_m,t_o)x_o h(t-(t_o+\tau_m)) \tag{6.30}$$

The initial condition predictor operates on (6.30) to produce an advanced version of the subplant initial condition response

$$n(t) = \Phi(t+\tau_c, t-\tau_m)m(t) = \Phi(t+\tau_c, t_o)x_o h(t-(t_o+\tau_m)) \tag{6.31}$$

which is added to (6.28), the advanced version of the forced subplant response, to produce an advanced version of the subplant output

$$p(t) = -(n(t)+q(t)) = -x(t+\tau_c)h(t-(t_o+\tau_m)) \tag{6.32}$$

The series controller operates on (6.32)

$$w(t) = -L(t+\tau_c)x(t+\tau_c)h(t-(t_o+\tau_m)) \tag{6.33}$$

and the zero function in the control delay yields the desired subplant input

$$u(t) = -L(t)x(t)h(t-(t_o+\tau_c+\tau_m)) \tag{6.34}$$

Implementing desired control (6.34), subplant equation (6.1) takes the form

$$\dot{x} = \{A(t)-B(t)L(t)h(t-(t_o+\tau_c+\tau_m))\}x(t), \quad x(t_o) = x_o \tag{6.35}$$

the solution of which is

$$x(t) = \begin{cases} \Phi(t,t_o)x_o & t_o \le t < t_o + \tau_c + \tau_m \\ \psi(t,t_o+\tau_c+\tau_m)\Phi(t_o+\tau_c+\tau_m,t_o)x_o & t_o+\tau_c+\tau_m \le t \le t_f \end{cases} \tag{6.36}$$

where $\psi(t,\sigma)$ is the transition matrix associated with $(A-BL)(t)$. From (6.34), the corresponding desired control is given by

$$u(t) = \begin{cases} 0 & t_o \le t < t_o + \tau_c + \tau_m \\ -L(t)\,\psi(t, t_o+\tau_c+\tau_m)\,\Phi(t_o+\tau_c+\tau_m, t_o)x_o & t_o+\tau_c+\tau_m \le t \le t_f \end{cases} \quad (6.37)$$

It is observed that when measurement delay is absent, Control scheme (3) remains optimal and Control scheme (4) reduces to optimal Control scheme (2). In this instance, Control scheme (2) is preferable to Control scheme (3) as it does not require prior knowledge of the initial state of the subplant. A comparison of the performances of Control schemes (3) and (4) is now undertaken. This will indicate whether the performance improvement achieved by Control scheme (3) outweighs the difficulties of obtaining and implementing prior knowledge of the initial state of the subplant.

6.3: A Comparison of the Performances of Control Schemes (3) and (4)

Control schemes (3) and (4) are compared when they incorporate a first-order, time-invariant subplant

$$\dot{x} = ax + bu; \quad a,b \neq 0; \quad x(0) = x_o \quad (6.38)$$

The associated time-invariant cost functional takes the form

$$J(u) = \int_0^\infty \{qx^2(\sigma) + ru^2(\sigma)\}d\sigma \quad (6.39)$$

where $q,r > 0$.

Consider firstly the cost of Control scheme (4) which comprises the sum of two parts

$$J(u) = J_1 + J_2(u) \quad (6.40)$$

The first term

$$J_1 = \int_0^{\tau_c+\tau_m} qx^2(\sigma)\,d\sigma \tag{6.41a}$$

where

$$x(t) = e^{at}x_o \qquad 0 \le t < \tau_c + \tau_m \tag{6.41b}$$

is the inevitable cost when the subplant input is zero. (In the current discussion both control schemes are matched. The subscripts 1 and 2 on the relevant costs bear no relation to any mismatched parameters.) The second term, the cost of control action

$$J_2 = \int_{\tau_c+\tau_m}^{\infty} \{qx^2(\sigma) + ru^2(\sigma)\}d\sigma \tag{6.42a}$$

is that of a LQP problem with initial conditions

$$x(\tau_c+\tau_m) = e^{a(\tau_c+\tau_m)}x_o \tag{6.42b}$$

Evaluating (6.41) and (6.42)

$$J_1 = \frac{qx_o^2}{2a}(e^{2a(\tau_c+\tau_m)}-1) \tag{6.43}$$

and

$$J_2 = \frac{r\ell}{b}\,e^{2a(\tau_c+\tau_m)}x_o^2 \tag{6.44a}$$

where

$$\ell = \frac{a+\sqrt{a^2+b^2q/r}}{b} \tag{6.44b}$$

Substituting (6.43) and (6.44) into (6.40), the cost of Control scheme (4) is given by

$$J(u) = \frac{qx_o^{\,2}}{2a}\,(e^{2a(\tau_c+\tau_m)}-1) + \frac{r\ell}{b}\,e^{2a(\tau_c+\tau_m)}\,x_o^2 \tag{6.45}$$

Now consider the cost of Control scheme (3). This also comprises the sum of an inevitable cost, C_1, and a controlled cost, $C_2(u)$

$$C(u) = C_1 + C_2(u) \tag{6.46}$$

For Control scheme (3) the subplant input is zero on $[0,\tau_c)$, in which case,

$$C_1 = \int_0^{\tau_c} qx^2(\sigma)\,d\sigma \tag{6.47a}$$

where

$$x(t) = e^{at}x_o \qquad 0 \le t < \tau_c \tag{6.47b}$$

The controlled cost

$$C_2(u) = \int_{\tau_c}^{\infty} \{qx^2(\sigma) + ru^2(\sigma)\}^2 d\sigma \tag{6.48a}$$

is that of a LQP problem with initial conditions

$$x(\tau_c) = e^{a\tau_c}x_o \tag{6.48b}$$

Comparing (6.46), (6.47) and (6.48) with (6.40), (6.41) and (6.42) respectively, the cost of Control scheme (3) is clearly obtained from (6.45) by setting $\tau_m = 0$

$$C(u) = \frac{qx_o^{\,2}}{2a}\,(e^{2a\tau_c}-1) + \frac{r\ell}{b}\,e^{2a\tau_c}x_o^{\,2} \tag{6.49}$$

The percentage improvement in performance arising from using Control scheme (3) rather than Control scheme (4) is defined using (6.45) and (6.49) as

$$I(a,\tau_m) = \left\{\frac{J(u)-C(u)}{J(u)}\right\} \times 100\% \qquad (6.50)$$

In explaining how (6.50) varies with measurement delay the cases of stable and unstable subplants will be considered separately. However, two facts are immediately clear; the cost of Control scheme (3) is independent of measurement delay, and an increase in measurement delay increases the inevitable cost of Control scheme (4).

Figure 6.4 shows that for a stable subplant the performance improvement via Control scheme (3) increases with measurement delay to a finite limit strictly less than 100%. The existence of this limit, which represents the maximum possible improvement, is explained with reference to Figure 6.5. Consider the subplant output and cost of Control scheme (4). As the measurement delay increases from zero, $x(\tau_c + \tau_m)$ decreases, producing a decrease in the cost of control action. However, the associated increase in the inevitable cost outweighs this decrease to produce an overall increase in cost. If the measurement delay is large enough the subplant output reaches zero along the path of the initial condition reponse. In this situation, Control scheme (4) incurs its greatest cost and (6.50) takes the limiting value.

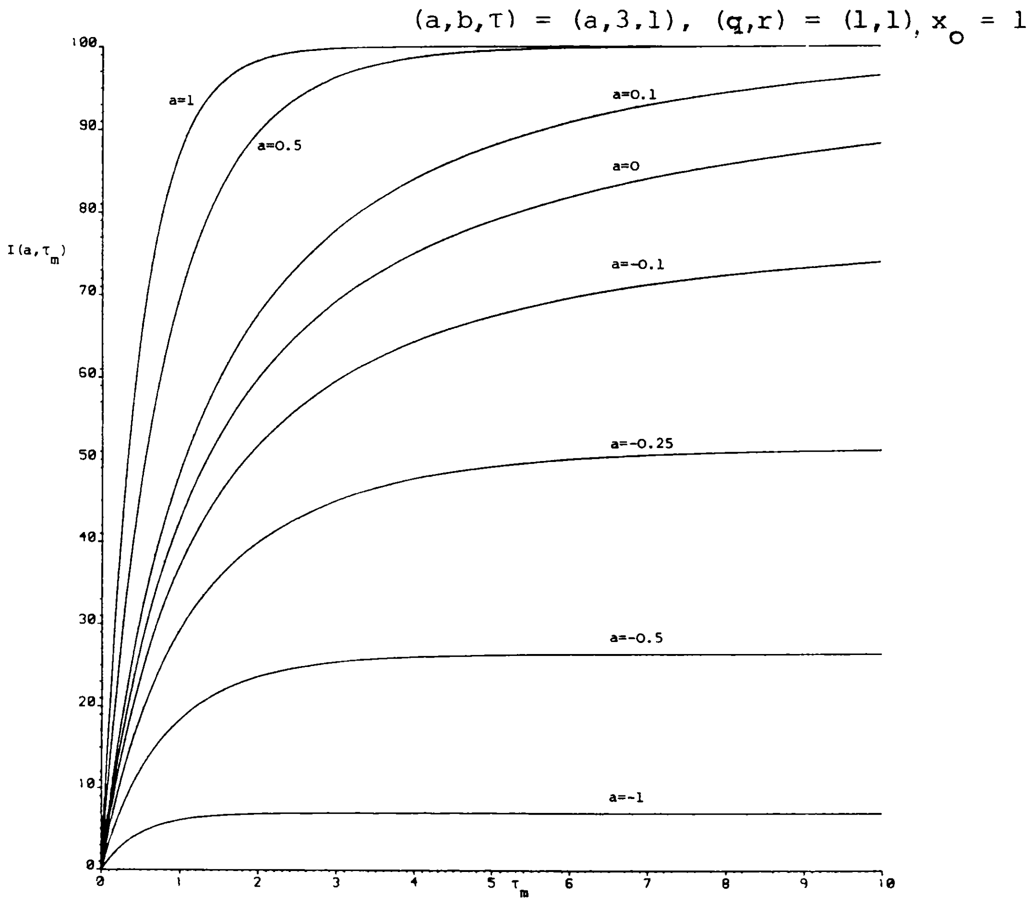


FIGURE 6.4: The Variation of Improvement via Control Scheme (3) with Measurement Delay.

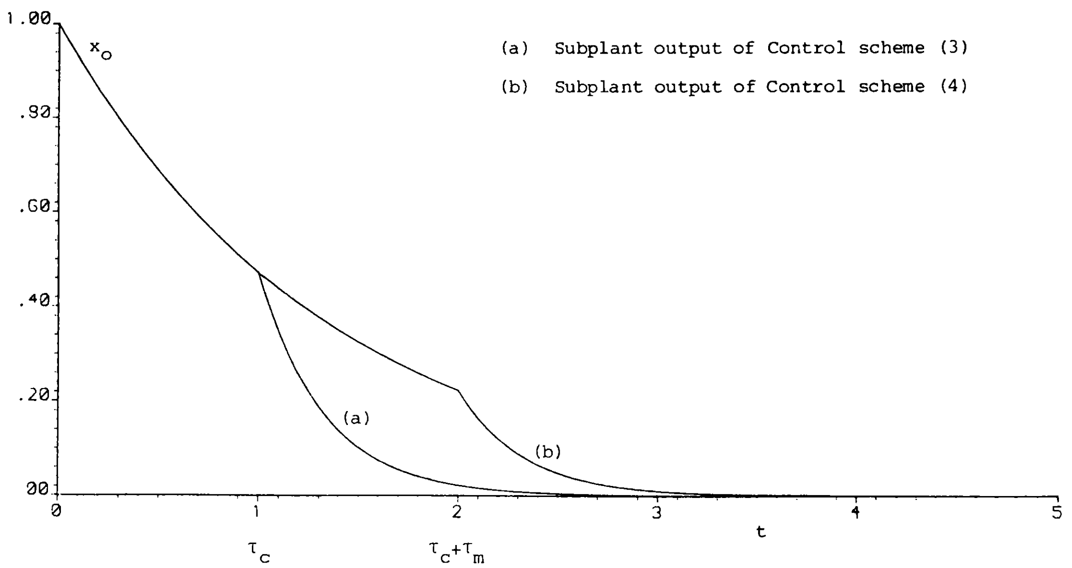


FIGURE 6.5: Stable Subplant Outputs for Control Schemes (3) and (4).

The maximum improvement available via Control scheme (3) can be found analytically by taking limits throughout (6.50)

$$\lim_{\tau_m \to \infty} I(a,\tau_m) = \left\{ 1 - \frac{C(u)}{\lim\limits_{\tau_m \to \infty} J(u)} \right\} \times 100\% \qquad (6.51)$$

As $a < 0$, it is clear that

$$\lim_{\tau_m \to \infty} J(u) = \frac{-qx_o^2}{2a} \qquad (6.52)$$

in which case, (6.51) reduces to

$$\lim_{\tau_m \to \infty} I(a,\tau_m) = \left\{ e^{2a\tau_c} \left(1 + \frac{2ar\ell}{bq}\right) \right\} \times 100\% \qquad (6.53)$$

Increasing parameter a (making parameter a less negative) makes the subplant less stable and ensures the initial condition response decays more slowly. Therefore, the greatest cost of Control scheme (4) (for variations in the measurement delay) is increased, as is the maximum improvement available via Control scheme (3), see Table 6.1.

a	Maximum Improvement via Control Scheme (3)
-1.0	7.03%
-0.5	26.34%
-0.25	51.35%
-0.1	76.59%

TABLE 6.1: Maximum Improvement via Control Scheme (3)

Figure 6.4 also shows that for an unstable subplant the performance improvement via Control scheme (3) increases with measurement delay to 100%. This is explained with reference to either Figure 6.6 or equations (6.43) and (6.44), which show that for Control scheme (4) increasing the measurement delay increases both the inevitable cost and the cost of control action. Therefore, for an unstable subplant, the cost of Control scheme (4) increases with measurement delay without limit, and the improvement available via Control scheme (3) takes the limiting value of 100%.

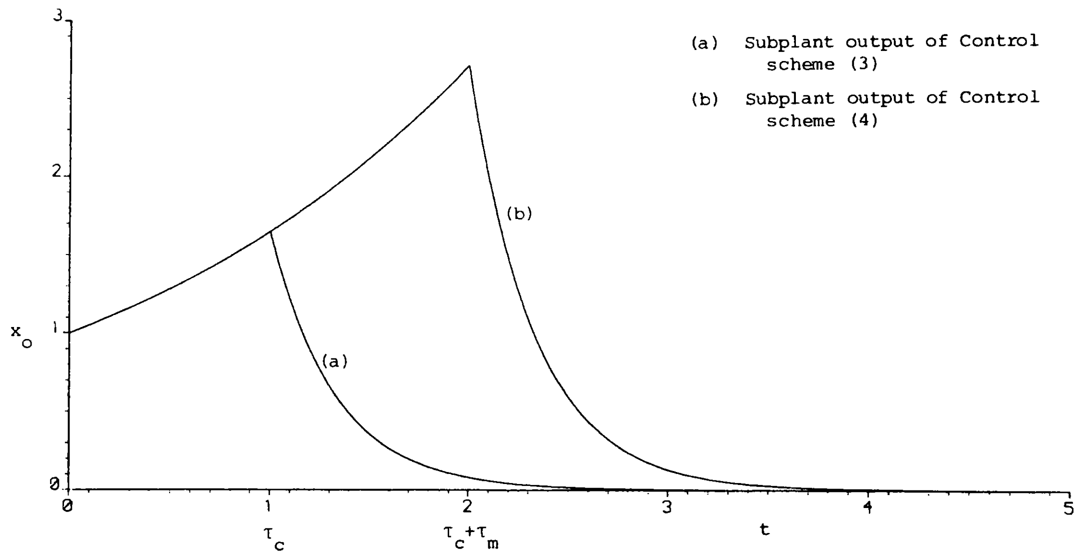


FIGURE 6.6: Unstable Subplant Outputs for Control Schemes (3) and (4)

Finally, it is noted from Figure 6.4 that for plant with a fixed measurement delay, the performance improvement available via Control scheme (3) is strictly increasing with parameter a. The main results of Section 6.3 may be summarised as follows. For a stable subplant, the performance improvement via Control scheme (3) increases with measurement delay to a finite limit depending on parameters a, b, τ_c, r and q, whereas for an unstable subplant, the improvement increases to 100%.

6.4: An Analysis of Mismatched Control Scheme (4)

As matched Control scheme (4) is suboptimal, mismatch may improve performance. Section 6.4 considers the types of mismatch which may produce this improvement. The investigation commences with a mathematical analysis of mismatched Control scheme (4) shown in Figure 6.7.

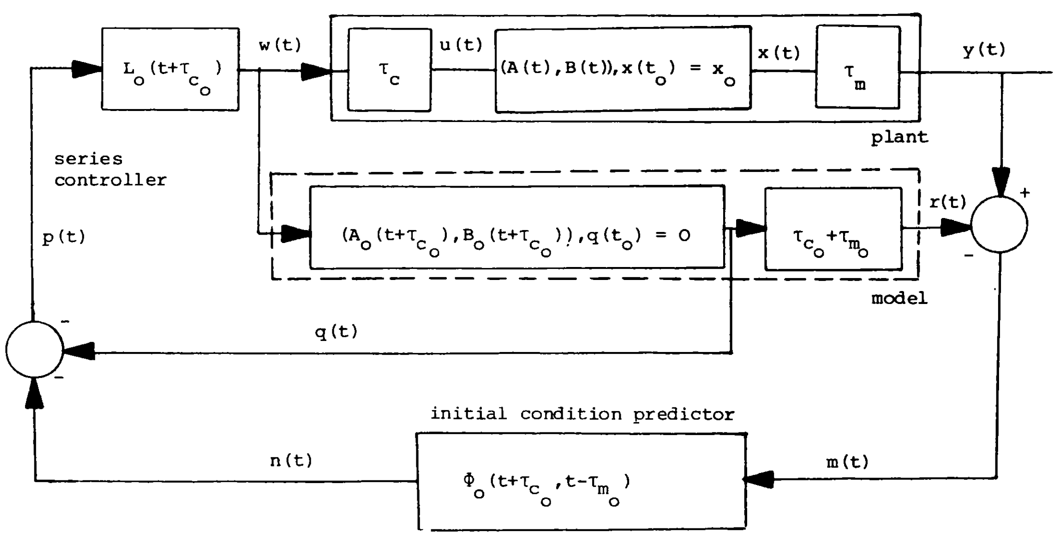


FIGURE 6.7: Mismatched Control Scheme (4)

As in the earlier chapters, it is assumed the model contains a linear system of the correct order, with a series time delay, but that precise parameter values may be unknown. The symbols $A_o(t)$, $B_o(t)$, τ_{c_o} and τ_{m_o} are the possibly mismatched models of $A(t)$, $B(t)$, τ_c and τ_m respectively, and it is assumed that $\tau_c + \tau_m$ is modelled by $\tau_{c_o} + \tau_{m_o}$. The series controller and initial condition predictor are calculated using the model values and denoted $L_o(t + \tau_{c_o})$ and $\Phi_o(t + \tau_{c_o}, t - \tau_{m_o})$ respectively. Now mismatch has been introduced, reference to Control scheme (4) should be interpreted as mismatched Control scheme (4). Likewise, for subplant input read mismatched subplant input.

Now mismatch appears in the argument of the model subplant, the initial condition predictor and the series controller, a certain amount of care must be exercised when considering the intervals for which these quantities are defined. As $A(t)$, $B(t)$, $\Phi(t,t_o)$ and $L(t)$ are defined on $[t_o,t_f]$ it is assumed that their models $A_o(t)$, $B_o(t)$, $\Phi_o(t,t_o)$ and $L_o(t)$ are defined on the same interval. As in the matched case, on the time interval $[t_o + \tau_m, t_f - \tau_c]$ the model subplant, the initial condition predictor and the series controller all produce functions which affect the cost functional. However, in this mismatched case, they are defined for time intervals with $t = t_f - \tau_{c_o}$ as the right endpoint.

This is no problem if $\tau_{c_o} \leqslant \tau_c$ as $t_f - \tau_c \leqslant t_f - \tau_{c_o}$ in which case, the elements of Control scheme (4) are only undefined for times when they do not contribute to the cost functional and an arbitrary definition can be made. However, if $\tau_c < \tau_{c_o}$, the model subplant, the initial condition predictor and the series controller are undefined on the interval

$(t_f - \tau_{c_o}, t_f - \tau_c]$ for which they contribute to the cost functional. In this instance, the relevant quantities are defined as constant over $(t_f - \tau_{c_o}, t_f - \tau_c]$, where the constant is their value at time $t = t_f - \tau_{c_o}$. Again for $t \in (t_f - \tau_c, t_f]$ an arbitrary definition is made.

A further consideration of this type occurs in the case of the initial condition predictor, which is only defined on the interval $[t_o + \tau_{m_o}, t_f - \tau_{c_o}]$. If $\tau_{m_o} \leqslant \tau_m$ then for $t \in [t_o + \tau_m, t_f - \tau_c]$, $t_o \leqslant t - \tau_{m_o}$ and $\Phi_o(.,t - \tau_{m_o})$ is defined. However, if $\tau_m < \tau_{m_o}$ then for $t \in [t_o + \tau_m, t_o + \tau_{m_o})$, $t - \tau_{m_o} < t_o$ and $\Phi_o(., t - \tau_{m_o})$ is undefined. In this instance, the initial condition predictor is defined as $\Phi_o(., t - \tau_{m_o}) = \Phi_o(., t_o)$ for $t \in [t_o + \tau_m, t_o + \tau_{m_o})$. In either case, if $\Phi_o(., t - \tau_{m_o})$ is undefined for any $t \in [t_o, t_o + \tau_m)$, it may be defined arbitrarily. Now the components of Control scheme (4) have been defined on the necessary intervals, an equation satisfied by the subplant input is determined by the following algebraic manipulation.

As in the matched case, the zero functions in the measurement and control delays ensure that

$$w(t) = 0 \qquad t_o \le t < t_o + \tau_m \tag{6.54}$$

and

$$u(t) = 0 \qquad t_o \le t < t_o + \tau_c + \tau_m \tag{6.55}$$

Again, any integral expression with $w(\sigma)$ or $u(\sigma)$ in the integrand is zero for σ on $[t_o, t_o + \tau_m)$ and $[t_o, t_o + \tau_c + \tau_m)$ respectively.

The subplant output is that of a time-varying linear system

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma \tag{6.56}$$

and zero function in the measurement delay ensures the plant output is given by

$$y(t) = x(t-\tau_m)h(t-(t_o+\tau_m)) \tag{6.57}$$

The delay-free model output is that of an advanced version of the mismatched linear system with zero initial state

$$\begin{aligned} q(t) &= \int_{t_o}^{t} \Phi_o(t+\tau_{c_o},\sigma+\tau_{c_o})B_o(\sigma+\tau_{c_o})w(\sigma)d\sigma \\ &= \int_{t_o}^{t} \Phi_o(t+\tau_{c_o},\sigma+\tau_{c_o})B_o(\sigma+\tau_{c_o})u(\sigma+\tau_c)d\sigma \\ &= \int_{t_o}^{t+\tau_c} \Phi_o(t+\tau_{c_o},\sigma+\tau_{c_o}-\tau_c)B_o(\sigma+\tau_{c_o}-\tau_c)u(\sigma)d\sigma \end{aligned} \tag{6.58}$$

From (6.58), the delayed model output is given by

$$r(t) = \int_{t_o}^{t+\tau_c-(\tau_{c_o}+\tau_{m_o})} \Phi_o(t-\tau_{m_o},\sigma+\tau_{c_o}-\tau_c)B_o(\sigma+\tau_{c_o}-\tau_c)u(\sigma)d\sigma \tag{6.59}$$

and subtracting (6.59) from (6.57) shows the term entering the outer feedback loop is given by

$$m(t) = y(t) - r(t) \tag{6.60}$$

The initial condition predictor operates on (6.60) to produce

$$\begin{aligned} n(t) &= \Phi_o(t+\tau_{c_o},t-\tau_{m_o})y(t) \\ &\quad -\int_{t_o}^{t+\tau_c-(\tau_{c_o}+\tau_{m_o})} \Phi_o(t+\tau_{c_o},\sigma+\tau_{c_o}-\tau_c)B_o(\sigma+\tau_{c_o}-\tau_c)u(\sigma)d\sigma \end{aligned} \tag{6.61}$$

and as q(t) and the integral term of n(t) have the same integrand, the addition of (6.58) and (6.61) yields

$$p(t) = -\left\{\Phi_o(t+\tau_{c_o}, t-\tau_{m_o})y(t) + \int_{t+\tau_c-(\tau_{c_o}+\tau_{m_o})}^{t+\tau_c} \Phi_o(t+\tau_{c_o}, \sigma+\tau_{c_o}-\tau_c)B_o(\sigma+\tau_{c_o}-\tau_c)u(\sigma)\,d\sigma\right\} \quad (6.62)$$

The series controller operates on (6.62)

$$w(t) = L_o(t+\tau_{c_o})p(t) \quad (6.63)$$

and the zero function in the control delay reveals the equation satisfied by the subplant input

$$u(t) = -L_o(t+\tau_{c_o}-\tau_c)\left\{\Phi_o(t+\tau_{c_o}-\tau_c, t-\tau_c-\tau_{m_o})x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m)) + \int_{t-(\tau_{c_o}+\tau_{m_o})}^{t} \Phi_o(t+\tau_{c_o}-\tau_c, \sigma+\tau_{c_o}-\tau_c)B_o(\sigma+\tau_{c_o}-\tau_c)u(\sigma)\,d\sigma\right\}$$
(6.64a)

where

$$x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m))$$
$$= \Phi(t-(\tau_c+\tau_m), t_o)x_o h(t-(t_o+\tau_c+\tau_m)) + \int_{t_o}^{t-(\tau_c+\tau_m)} \Phi(t-(\tau_c+\tau_m), \sigma)B(\sigma)u(\sigma)\,d\sigma$$
(6.64b)

A concept referred to as equivalence is now introduced, to determine the types of mismatch that may improve Control scheme (4). Two cases of Control scheme (4) are considered; the first incorporating a plant with delay $\tau_c + \tau_m$ in control, and the second incorporating a plant with delays τ_c in control and τ_m in measurement. The two cases are defined as equivalent if they produce the same subplant input, in which case, their subplant outputs and in particular their cost functionals are equal.

When measurement delay is absent, Control scheme (4) reduces to Control scheme (2) which is optimal when matched. As mismatch impairs the performance of Control scheme (4) in the first case, it will impair the performance in the second case if the two cases are equivalent. The forms of mismatch which allow equivalence are now determined, in the knowledge that such forms cannot improve the performance of Control scheme (4) for plants with delays in control and measurement.

The subplant input for a plant with delay τ_c in control and no measurement delay is the solution of the equation formed from (6.64) when $\tau_m = \tau_{m_o} = 0$

$$u(t) = -L_o(t-\tau_c+\tau_{c_o})\Big\{\Phi_o(t-\tau_c+\tau_{c_o}, t-\tau_c)x(t-\tau_c)h(t-(t_o+\tau_c))$$

$$+ \int_{t-\tau_{c_o}}^{t} \Phi_o(t-\tau_c+\tau_{c_o}, \sigma-\tau_c+\tau_{c_o})B_o(\sigma-\tau_c+\tau_{c_o})u(\sigma)\,d\sigma\Big\} \qquad (6.65a)$$

where

$$x(t-\tau_c)h(t-(t_o+\tau_c))$$

$$= \Phi(t-\tau_c,t_o)x_o h(t-(t_o+\tau_c)) + \int_{t_o}^{t-\tau_c} \Phi(t-\tau_c,\sigma)B(\sigma)u(\sigma)\,d\sigma \qquad (6.65b)$$

The subplant input for a plant with delay $\tau_c + \tau_m$ in control and no measurement delay, is the solution of the equation formed from (6.65) when τ_c and τ_{c_o} are replaced by $\tau_c + \tau_m$ and $\tau_{c_o} + \tau_{m_o}$ respectively

$$u(t) = -L_o(t-(\tau_c+\tau_m) + (\tau_{c_o}+\tau_{m_o}))$$

$$\Big\{\Phi_o(t-(\tau_c+\tau_m)+(\tau_{c_o}+\tau_{m_o}), t-(\tau_c+\tau_m))x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m))$$

$$+ \int_{t-(\tau_{c_o}+\tau_{m_o})}^{t} \Phi_o(t-(\tau_c+\tau_m)+(\tau_{c_o}+\tau_{m_o}), \sigma-(\tau_c+\tau_m)+(\tau_{c_o}+\tau_{m_o}))$$

$$\times B_o(\sigma-(\tau_c+\tau_m)+(\tau_{c_o}+\tau_{m_o}))u(\sigma)\,d\sigma\Big\} \qquad (6.66a)$$

where

$$x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m))$$

$$= \Phi(t-(\tau_c+\tau_m),t_o)x_o h(t-(t_o+\tau_c+\tau_m)) + \int_{t_o}^{t-(\tau_c+\tau_m)} \Phi(t-(\tau_c+\tau_m),\sigma)B(\sigma)u(\sigma)d\sigma \qquad (6.66b)$$

As equations (6.64) and (6.66) are not identical, the two cases of Control scheme (4) may not be equivalent for a time-varying subplant in the presence of mismatch in all parameter simultaneously.

However, when $\tau_m = \tau_{m_o}$, that is when mismatch in measurement delay is absent, both (6.64) and (6.66) reduce to

$$u(t) = -L_o(t-\tau_c+\tau_{c_o})\Big\{\Phi_o(t-\tau_c+\tau_{c_o},t-(\tau_c+\tau_m))x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m))$$

$$+ \int_{t-(\tau_{c_o}+\tau_m)}^{t} \Phi_o(t-\tau_c+\tau_{c_o},\sigma-\tau_c+\tau_{c_o})B_o(\sigma-\tau_c+\tau_{c_o})u(\sigma)d\sigma\Big\} \qquad (6.67a)$$

where

$$x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m))$$

$$= \Phi(t-(\tau_c+\tau_m),t_o)x_o h(t-(t_o+\tau_c+\tau_m)) + \int_{t_o}^{t-(\tau_c+\tau_m)} \Phi(t-(\tau_c+\tau_m),\sigma)B(\sigma)u(\sigma)d\sigma \qquad (6.67b)$$

Therefore, for a time-varying subplant the two cases of Control scheme (4) are equivalent if mismatch in measurement delay is absent.

Now consider a time-invariant subplant; equations (6.64) and (6.66) reduce to (6.68) and (6.69) respectively

$$u(t) = -L_o(t-\tau_c+\tau_{c_o})\Big\{e^{A_o(\tau_{c_o}+\tau_{m_o})}x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m))$$

$$+ \int_{t-(\tau_{c_o}+\tau_{m_o})}^{t} e^{A_o(t-\sigma)}B_o u(\sigma)d\sigma\Big\} \qquad (6.68a)$$

where

$$x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m)) =$$

$$e^{A(t-(\tau_c+\tau_m)-t_o)} x_o h(t-(t_o+\tau_c+\tau_m)) + \int_{t_o}^{t-(\tau_c+\tau_m)} e^{A(t-(\tau_c+\tau_m)-\sigma)} \cdot$$

$$Bu(\sigma)d\sigma \tag{6.68b}$$

and

$$u(t) = -L_o(t-(\tau_c+\tau_m)+(\tau_{c_o}+\tau_{m_o}))\left\{e^{A_o(\tau_{c_o}+\tau_{m_o})} x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m)) + \int_{t-(\tau_{c_o}+\tau_{m_o})}^{t} e^{A_o(t-\sigma)} B_o u(\sigma)d\sigma\right\} \tag{6.69}$$

where $x(t-(\tau_c+\tau_m))h(t-(t_o+\tau_c+\tau_m))$ is given by (6.68b). The difference between (6.68) and (6.69) results from the presence of mismatched measurement delay in the argument of the series controller. When the LQP problem is over the semi-infinite time horizon, the series controller is constant and the two cases of Control scheme (4) are equivalent in the presence of mismatch in all parameters simultaniously.

The results of Section 6.4 may be summarised as follows. The two cases of Control scheme (4) are equivalent if mismatch in measurement delay is absent. Furthermore, in the presence of mismatch in measurement delay, the two cases of Control scheme (4) are equivalent for a time-invariant subplant, with a semi-infinite time horizon cost functional. In other words, only mismatch in measurement delay may improve the performance of Control scheme (4) for plants with delays in control and measurement. Moreover, if the subplant is time-invariant with a semi-infinite time horizon cost functional no improvement by mismatch is possible.

Conclusions

Two matched control schemes are presented to minimise a quadratic cost functional for a time-varying plant with delays in control and measurement. Matched Control scheme (3) is optimal but requires prior knowledge of the initial state of the subplant to determine an initial state for the model. Control scheme (4) is a natural extension of Control scheme (2). Although matched Control scheme (4) is suboptimal it does not require prior knowledge of the initial state of the subplant as the initial state of the model is chosen to be zero.

For a stable subplant, the performance improvement available from using matched Control scheme (3), rather than matched Control scheme (4), increases with measurement delay to a finite limit which depends on the plant and cost functional parameters. Furthermore, for an unstable subplant, the improvement available via matched Control scheme (3) increases with measurement delay to 100%. As matched Control scheme (4) is suboptimal, mismatch may improve performance. Using the concept of equivalence, it is shown that mismatch in any parameter other than measurement delay will impair performance. Furthermore, for a time-invariant subplant with semi-infinite time horizon cost functional, mismatch in measurement delay will also impair performance.

CONCLUSIONS

A common method of controlling plants with series time delays is to implement a predictor control scheme, a feature of which is the use of a plant model. The aim of this thesis is to examine the effects of mismatch on the stability and performance of predictor control schemes.

The major part of the current mismatch literature centres around the Smith control scheme. This is a control scheme which when matched, removes the time delay from the feedback loop by producing a delayed version of a parametrically optimised delay-free response. The externalising of the delay reduces any design to that of a delay-free scheme, which allows the use of higher gains and in turn produces a faster response.

The mismatch literature shows it is general engineering practice to overestimate an unknown time delay, as this strategy may produce an improved response. This is consistent with the fact that minimising the cost of the Smith scheme is a parametric optimisation problem, and the presence of mismatch serves to introduce extra parameters which may be exploited to produce performance improvement. An investigation into the benefits of overestimation shows the error of the mismatched Smith scheme is an infinite series of terms, the first of which is the matched error. Improvement by mismatch occurs when an appropriate choice of model delay allows the first term to be cancelled by subsequent terms. This produces a mismatched error "close" to the zero function, which ensures the mismatched Smith scheme a reduced cost.

The optimal model delay producing the best cancellation may be obtained numerically, however, optimisation routines for time delay schemes are expensive in terms of computer time. An inexpensive technique to estimate the optimal model delay is developed, involving the simulation of two delay-free curves. This technique leads to the conjecture that improvement is by underestimation or overestimation of the plant delay depending on whether mismatch is being utilised to cancel a peak or a trough of the matched error. Examples of both methods of improvement are given. Furthermore, the amount of improvement achieved is determined by the size of the peak or trough being cancelled.

In several papers supporting overestimation of the time delay the subplants considered are of second-order. The significant feature of the error of a second-order subplant is that the first trough is greater than any subsequent peak or trough. Therefore, improvement is most readily obtained when the plant delay is such that this first trough is cancelled, which gives insight into why these papers favour the practice of over-estimating time delays.

As the potential for improvement by mismatch depends on the size of the plant delay, the addition of time delays into the Smith scheme can be utilised to produce further improvement by mismatch. Similarly, the performance of delay-free schemes can be improved by the addition of an outer feedback loop containing two time delays. The optimal values of the two delays are determined from the earlier analysis of associated time delay schemes.

When a quadratic cost functional is associated with a synthesis procedure for a time delay plant a matched predictor control scheme is optimal. Matched Control scheme (1) incorporating a plant with a control time delay and a time-invariant subplant, optimises a time-invariant quad-ratic cost functional over the semi-infinite time horizon. Control scheme (2) is its time-varying extension. The essence of both control schemes is the use of a perfectly matched plant model to produce an advanced version of the state. However, in most practical situations the plant will not be known exactly, which motivates a study into the effects of mismatch on Control scheme (1).

A mathematical analysis of mismatched Control scheme (1) yields matrix integral equations satisfied by the mismatched subplant input and output. The solution of these equations is straightforward in the first-order case, with the result that both mismatched input and output are infinite series with terms consisting of delays, exponential factors and factors which are powers of t. For mismatch in the subplant parameters the n^{th} term of the series contains delay $n\tau$, whereas for mismatch in delay, the n^{th} term is a sum of n subterms, the i^{th} of which contains the delay $(n-i)\tau + i\tau_o$, $0 \leqslant i \leqslant n-1$. The mismatched input and output reduce to their matched forms when mismatch is absent and to their anticipated forms on $[0,\tau]$. Furthermore, the mismatched input and output are continuous for $t \neq \tau$ and $t \geqslant 0$ respectively. A numerical and analytical examination of the expressions for the mismatched input and output determines how mismatch affects the stability and performance of Control scheme (1) for a representative selection of first-order subplants.

A control scheme is defined as stable if it produces a bounded output and the subplant is stable if all its eigenvalues lie in the left-half plane. Matched Control scheme (1) is stable regardless of the stability of the subplant, as its output optimises a LQP problem. When Control scheme (1) incorporates a stable subplant it is stable for any value of mismatch in b and any realisable model delay. However, Control scheme (1) with a stable subplant can be made unstable when mismatch in parameter a creates a sufficiently unstable model. When Control scheme (1) incorporates an unstable subplant, it is only stable for a closed, finite, non-symmetric interval of model values about the matched value.

The performance of Control scheme (1) is optimised when plant and model are identical. When Control scheme (1) incorporates a stable subplant, if the matched values are not available, the best method of restricting the increase in cost is to underestimate parameter a and overestimate the delay. When Control scheme (1) incorporates an unstable subplant, as mismatch increases the cost diverges to infinity.

When the plant also contains a measurement delay, optimal matched Control scheme (3) minimising a quadratic cost functional requires prior knowledge of the initial state of the subplant. Control scheme (4) is presented as an alternative. Although matched Control scheme (4) is suboptimal it has the advantage of not requiring prior knowledge of the initial state of the subplant. For a stable subplant, the performance improvement available from using Control scheme (3), rather than Control scheme (4), increases with measurement delay to a finite limit which depends on the plant and cost functional parameters. For an unstable subplant, the improvement available via Control scheme (3) increases with measurement delay to 100%. As matched Control scheme (4) is suboptimal, mismatch may improve performance. Using the concept of equivalence, it is shown that mismatch in any parameter other than the measurement delay will impair performance. Furthermore, for a time-invariant subplant with a semi-infinite time horizon cost functional, mismatch in measurement delay will also impair performance.

Finally, some ideas for further research. It is recalled that for the first-order example of Chapter 2, if the plant delay is small enough, the ISE cost functional is optimised by a zero model delay. In this instance, the Laplace domain expression of the cost functional, obtained using Parseval's theorem (Jacobs, 1974), contains only the plant delay and the necessary contour integral may be computed to obtain a closed form solution (Walton and Marshall, to appear). This closed form is minimised with respect to the plant delay by the value of $\pi/6$, which compares well with the numerical result of $\tau^* = 0.52$. Research is currently being undertaken into obtaining closed form solutions for the cost functionals of higher-order examples with non-zero model delays. Gorecki & Popek (1983) have some relevant results on this topic.

The studies of Chapters 4 and 5 only consider mismatch in individual parameters. This work could be extended by obtaining expressions for the subplant input and output in the presence of mismatch in several parameters. An analysis of these expressions would reveal any interaction between the different types of mismatch and answer questions as to whether the effects of mismatch in one parameter can be minimised by mismatch in another. In other words, if mismatch in a particular parameter is unavoidable, and the control scheme is therefore suboptimal, is the best strategy to match the remaining parameters? Another extension in this area of work is to obtain results for higher-order examples.

The comparison of Control schemes (3) and (4) is undertaken when both schemes are matched. Following an analysis of mismatched Control scheme (3), the performances of the two schemes could be compared in the presence of the different types of mismatch. Furthermore, if prior knowledge of the initial state of the subplant is not available, rather than resort to Control scheme (4), Control scheme (3) may be implemented using an estimate of the subplant initial state. The effects on Control scheme (3) of mismatch in the subplant initial state could then be investigated. Finally, the majority of the work in this thesis may be reappraised when permitting various types of noise, non-zero initial functions in the time delays, and output feedback rather than state feedback.

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APPENDIX

APPENDIX

A.1 The Form and Properties of the Subplant Output for Mismatch in a

A.1.1 The Form of the Output

For mismatch in a, the Laplace transform of the output is given by

$$\bar{x}(s) = \frac{x_o}{s-a} - \frac{bk_1 x_o (s-a_o) e^{-s\tau}}{(s-a)((s-a)(s-c_1) - k_1 d_1 e^{-s\tau})} \tag{A.1a}$$

where

$$c_1 = a_o - \ell_1 b \;, \quad k_1 = \ell_1 e^{a_o \tau} \;, \quad \ell_1 = \frac{a_o + \sqrt{a_o^2 + b^2 q/r}}{b} \tag{A.1b}$$

and

$$d_1 = b(a_o - a) \tag{A.1c}$$

For $s \in \mathbb{C}$ with real part large enough

$$\left| \frac{k_1 d_1 \; e^{-s\tau}}{(s-a)(s-c_1)} \right| < 1 \tag{A.2}$$

and (A.1a) may be expressed as a geometric series

$$\frac{bk_1 x_o (s-a_o) \; e^{-s\tau}}{(s-a)((s-a)(s-c_1) - k_1 d_1 \; e^{-s\tau})}$$

$$= \frac{bk_1 x_o (s-a_o) e^{-s\tau}}{(s-a) \; (s-c_1) \left\{ 1 - \dfrac{k_1 d_1 \; e^{-s\tau}}{(s-a)(s-c_1)} \right\}}$$

$$= bk_1x_o(s-a_o) \sum_{n=1}^{\infty} \frac{(k_1d_1)^{n-1} e^{-sn\tau}}{(s-a)^{n+1}(s-c_1)^n} \tag{A.3}$$

Now, expanding by partial fractions

$$\frac{1}{(s-a)^{n+1}(s-c_1)^n}$$

$$= \sum_{i=o}^{n} p(n,i) \frac{1}{(a-c_1)^{n+i}} \frac{1}{(s-a)^{n-i+1}}$$

$$+ \sum_{i=o}^{n-1} p(n+1,i) \frac{1}{(c_1-a)^{n+i+1}} \frac{1}{(s-c_1)^{n-i}} \tag{A.4}$$

where

$$p(m,o) = 1 \ , \quad p(m,i) = \frac{m(m+1)\ldots(m+i-1)}{i!} \qquad 1 \leqslant i \leqslant m-1 \tag{A.5}$$

Adopting the notation

$$\mathcal{L}^{-1} \frac{(k_1d_1)^{n-1}}{(s-a)^{n+1}(s-c_1)^n} = g_n(t) \tag{A.6}$$

the inverse Laplace transform of (A.4) shows

$$g_n(t) = (k_1d_1)^{n-1} \left\{ \sum_{i=o}^{n} \hat{g}_{n_i}(t) + \sum_{i=o}^{n-1} \tilde{g}_{n_i}(t) \right\} \tag{A.7}$$

where

$$\hat{g}_{n_i}(t) = p(n,i) \frac{1}{(a-c_1)^{n+i}} \frac{t^{n-i}}{(n-i)!} e^{at} \tag{A.8}$$

and

$$\tilde{g}_{n_i}(t) = p(n+1,i)\,\frac{1}{(c_1-a)^{n+i+1}}\;\frac{t^{n-i-1}}{(n-i-1)!}\;e^{c_1 t} \qquad (A.9)$$

Furthermore, substituting $t = 0$ into (A.7), (A.8) and (A.9) reveals

$$g_n(0) = (k_1d_1)^{n-1}\left\{\hat{g}_{n_n}(0) + g_{n_{n-1}}(0)\right\}$$

$$= (k_1d_1)^{n-1}\left\{p(n,n) + p(n+1,n-1)\right\}\frac{1}{(a-c_1)^{2n}} = 0 \qquad (A.10)$$

in which case

$$\mathcal{L}^{-1}\;\frac{s(k_1d_1)^{n-1}}{(s-a)^{n+1}(s-c_1)^n} = g_n'(t) \qquad (A.11)$$

Applying these results, the inverse Laplace transform of (A.3) shows the output for mismatch in a is given by

$$x(t) = e^{at}x_o - bk_1x_o\sum_{n=1}^{\infty}\{g_n'(t-n\tau) - a_o\,g_n(t-n\tau)\}\,h(t-n\tau) \qquad (A.12a)$$

where

$$g_n(t) = (k_1d_1)^{n-1}\left\{\sum_{i=o}^{n}\hat{g}_{n_i}(t) + \sum_{i=o}^{n-1}\tilde{g}_{n_i}(t)\right\} \qquad (A.12b)$$

$$\hat{g}_{n_i}(t) = p(n,i)\,\frac{1}{(a-c_1)^{n+i}}\;\frac{t^{n-1}}{(n-i)!}\;e^{at} \qquad (A.12c)$$

$$\tilde{g}_{n_i}(t) = p(n+1,i) \frac{1}{(c_1-a)^{n+i+1}} \frac{t^{n-i-1}}{(n-i-1)!} e^{c_1 t} \qquad \text{(A.12d)}$$

$$p(m,o) = 1 , \quad p(m,i) = \frac{(-1)^i m(m+1)\ldots(m+i-1)}{i!} \quad 1 \leqslant i \leqslant m-1 \qquad \text{(A.12e)}$$

$$c_1 = a_o - \ell_1 b , \quad k_1 = \ell_1 e^{a_o \tau} , \quad \ell_1 = \frac{a_o + \sqrt{a_o^{\,2}+b^2 q/r}}{b} \qquad \text{(A.12f)}$$

and

$$d_1 = b(a_o - a) \qquad \text{(A.12g)}$$

A.1.2 The Reduction of the Output to Matched Form

When plant and model match $a_o = a$ and the parameters of (A.12f) and (A.12g) satisfy

$$d_1 = d = 0 , \quad \ell_1 = \ell , \quad k_1 = k , \quad c_1 = c \qquad \text{(A.13)}$$

The presence of zero valued d_1 in (A.12b) ensures

$$g_n(t) = 0 , \quad g'_n(t) = 0 \qquad n = 2,3,\ldots \qquad \text{(A.14)}$$

in which case (A.12a) reduces to

$$x(t) = e^{at}x_o - bkx_o \{g'_1(t-\tau) - ag_1(t-\tau)\} h(t-\tau) \qquad \text{(A.15)}$$

Examining the terms of (A.15) in detail

$$g_1(t-\tau) = \hat{g}_{1_o}(t-\tau) + \hat{g}_{1_1}(t-\tau) + \tilde{g}_{1_o}(t-\tau)$$

$$= \frac{(t-\tau)e^{a(t-\tau)}}{a-c} - \frac{e^{a(t-\tau)}}{(a-c)^2} + \frac{e^{c(t-\tau)}}{(c-a)^2} \qquad (A.16)$$

and

$$g_1'(t-\tau) = \frac{(t-\tau)ae^{a(t-\tau)} + e^{a(t-\tau)}}{a-c}$$

$$- \frac{ae^{a(t-\tau)}}{(a-c)^2} + \frac{c\ e^{c(t-\tau)}}{(c-a)^2} \qquad (A.17)$$

Combining (A.16) and (A.17)

$$g_1'(t-\tau) - ag_1(t-\tau)$$

$$= \left\{\frac{a(t-\tau)e^{a(t-\tau)} + e^{a(t-\tau)}}{b\ell} - \frac{ae^{a(t-\tau)}}{(b\ell)^2} + \frac{(a-\ell b)e^{(a-\ell b)(t-\tau)}}{(b\ell)^2}\right\}$$

$$- \left\{\frac{a(t-\tau)e^{a(t-\tau)}}{b\ell} - \frac{ae^{a(t-\tau)}}{(b\ell)^2} + \frac{ae^{(a-\ell b)(t-\tau)}}{(b\ell)^2}\right\}$$

$$= \frac{e^{a(t-\tau)} - e^{(a-\ell b)(t-\tau)}}{b\ell} \qquad (A.18)$$

Substituting (A.18) into (A.15), mismatched output (A.12a) reduces to

$$x(t) = e^{at}x_o - e^{a\tau}x_o\ (e^{a(t-\tau)} - e^{(a-\ell b)(t-\tau)})\ h(t-\tau)$$

$$= e^{at}x_o - (e^{at}x_o - e^{(a-\ell b)(t-\tau)}e^{a\tau}x_o)\ h(t-\tau) \qquad (A.19)$$

an alternative presentation of which is

$$x(t) = \begin{cases} e^{at}x_o & 0 \leqslant t < \tau \\ e^{(a-\ell b)(t-\tau)}e^{a\tau}x_o & t \geqslant \tau \end{cases} \tag{A.20}$$

the desired first-order version of matched output (3.29).

A.1.3 The Output over [0,τ]

When $t < \tau$, by the definition of the Heaviside step function

$$h(t-n\tau) = 0 \qquad n = 1,2,\ldots \tag{A.21}$$

and (A.12a) reduces to

$$x(t) = e^{at}x_o \qquad 0 \leqslant t < \tau \tag{A.22}$$

It therefore remains to consider the case of $t = \tau$. In this instance

$$h(t-n\tau) = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,\ldots \end{cases} \tag{A.23}$$

which ensures (A.12a) satisfies

$$x(\tau) = e^{a\tau}x_o - bk_1x_o\{g_1'(0) - a_og_1(0)\} \tag{A.24}$$

Examining the terms of (A.24) in detail

$$g_1(t) = \hat{g}_{1_o}(t) + \hat{g}_{1_1}(t) + \tilde{g}_{1_o}(t)$$

$$= \frac{te^{at}}{a-c_1} - \frac{e^{at}}{(a-c_1)^2} + \frac{e^{c_1 t}}{(c_1-a)^2} \tag{A.25}$$

and

$$g_1'(t) = \frac{tae^{at} + e^{at}}{a-c_1} - \frac{ae^{at}}{(a-c_1)^2} + \frac{c_1 e^{c_1 t}}{(c_1-a)^2} \tag{A.26}$$

When $t = 0$, (A.25) and (A.26) become

$$g_1(0) = \frac{-1}{(a-c_1)^2} + \frac{1}{(c_1-a)^2} = 0 \tag{A.27}$$

and

$$g_1'(0) = \frac{1}{a-c_1} - \frac{a}{(a-c_1)^2} + \frac{c_1}{(c_1-a)^2} = 0 \tag{A.28}$$

Substituting (A.27) and (A.28) into (A.24) shows that for the case $t = \tau$ (A.12a) reduces to

$$x(\tau) = e^{a\tau} x_o \tag{A.29}$$

Combining (A.22) and (A.29)

$$x(t) = e^{at} x_o \qquad 0 \leqslant t \leqslant \tau \tag{A.30}$$

the expected form of the mismatched output over $[0,\tau]$.

A.1.4 The Continuity of the Output

As the input in the presence of mismatch in a is continuous for $t \neq \tau$, so is (A.12) the corresponding output. Furthermore, as (A.29) shows (A.12) to be continuous for $t = \tau$, (A.12) is continuous for $t \geqslant 0$. Continuity of (A.12) could be shown directly, following the style adopted in Section 4.2.4 for the mismatched input.

A.2 The Form and Properties of the Subplant Input and Output for Mismatch in b

A.2.1 The Form of the Input

For mismatch in b, the Laplace transform of the input is given by

$$\bar{u}(s) = \frac{-k_2 x_o e^{-s\tau}}{(s-c_2) - k_2 d_2 e^{-s\tau}} \qquad \text{(A.31a)}$$

where

$$c_2 = a-\ell_2 b_o\,, \quad k_2 = \ell_2 e^{a\tau}\,, \quad \ell_2 = \frac{a + \sqrt{a^2+b_o{}^2 q/r}}{b_o} \qquad \text{(A.31b)}$$

and

$$d_2 = b_o - b \qquad \text{(A.31c)}$$

For $s \in \mathbb{C}$ with real part large enough

$$\left| \frac{k_2 d_2 \, e^{-s\tau}}{s-c_2} \right| < 1 \tag{A.32}$$

and (A.31a) may be expressed as a geometric series

$$\bar{u}(s) = \frac{-k_2 x_o \, e^{-s\tau}}{(s-c_2)\left\{1 - \dfrac{k_2 d_2 \, e^{-s\tau}}{(s-c_2)}\right\}}$$

$$= - k_2 x_o \sum_{n=1}^{\infty} \frac{(k_2 d_2)^{n-1} \, e^{-sn\tau}}{(s-c_2)^n} \tag{A.33}$$

Adopting the notation

$$\mathcal{L}^{-1} \frac{(k_2 d_2)^{n-1}}{(s-c_2)^n} = f_n(t) \tag{A.34}$$

the inverse Laplace transform of (A.33) shows the input for mismatch in b is given by

$$u(t) = - k_2 x_o \sum_{n=1}^{\infty} f_n(t-n\tau) \, h(t-n\tau) \tag{A.35a}$$

where

$$f_n(t) = (k_2 d_2)^{n-1} \frac{t^{n-1}}{(n-1)!} \, e^{c_2 t} \tag{A.35b}$$

$$c_2 = a - \ell_2 b_o \, , \quad k_2 = \ell_2 e^{a\tau} \, , \quad \ell_2 = \frac{a + \sqrt{a^2 + b_o{}^2 q/r}}{b_o} \tag{A.35c}$$

and

$$d_2 = b_o - b \tag{A.35d}$$

A.2.2 The Reduction of the Input to Matched Form

When plant and model match $b_o = b$ and the parameters of (A.35c) and (A.35d) satisfy

$$d_2 = d = 0 \ , \quad \ell_2 = \ell \ , \quad k_2 = k \ , \quad c_2 = c \tag{A.36}$$

The terms of (A.35b) containing d_2 are therefore zero

$$f_n(t) = 0 \ , \qquad n = 2,3,\ldots \tag{A.37}$$

in which case, (A.35a) reduces to

$$u(t) = -\, kx_o f_1(t-\tau)\ h(t-\tau) \tag{A.38}$$

From (A.35b)

$$f_1(t-\tau) = e^{(a-\ell b)(t-\tau)} \tag{A.39}$$

and substituting (A.39) into (A.38) mismatched input (A.35a) reduces to

$$u(t) = -\, \ell\ e^{(a-\ell b)(t-\tau)}\ e^{a\tau} x_o\ h(t-\tau) \tag{A.40}$$

the desired first-order version of matched input (3.30).

A.2.3 The Input over $[0,\tau]$

When $t < \tau$, by the definition of the Heaviside step function

$$h(t-n\tau) = 0 \quad n = 1,2,\ldots \tag{A.41}$$

and (A.35a) reduces to

$$u(t) = 0 \qquad 0 \leqslant t < \tau \tag{A.42}$$

It therefore remains to consider the case of $t = \tau$.

In this instance

$$h(t-n\tau) = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,\ldots \end{cases} \tag{A.43}$$

which ensures (A.35a) satisfies

$$\begin{aligned} u(\tau) &= - k_1 x_o f_1(0) \\ &= - \ell_2 \, e^{a\tau} \, x_o \end{aligned} \tag{A.44}$$

Combining (A.42) and (A.44)

$$u(t) = \begin{cases} 0 & 0 \leqslant t < \tau \\ -\ell_2 e^{a\tau} x_o & t = \tau \end{cases} \tag{A.45}$$

the expected form over $[0,\tau]$.

A.2.4 The Continuity of the Input

The mismatched input is clearly continuous for $t < \tau$ and discontinuous when $t = \tau$. Continuity for $t > \tau$ follows from (A.35b) as

$$f_n(0) = 0 \qquad n = 2,3,\ldots \tag{A.46}$$

In other words, $f_n(t-n\tau)$ is zero when it first contributes to (A.35) at time $t = n\tau$. This shows the mismatched input changes smoothly between adjoining time intervals and is continuous for $t \neq \tau$.

A.2.5 The Form of the Output

For mismatch in b , the Laplace transform of the output is given by

$$\bar{x}(s) = \frac{x_o}{s-a} - \frac{bk_2x_oe^{-s\tau}}{(s-a)((s-c_2) - k_2d_2e^{-s\tau})} \tag{A.47a}$$

where

$$c_2 = a - \ell_2b_o \ , \quad k_2 = \ell_2e^{a\tau}, \quad \ell_2 = \frac{a + \sqrt{a^2+b_o{}^2q/r}}{b_o} \tag{A.47b}$$

and

$$d_2 = b_o - b \tag{A.47c}$$

For $s \in \mathbb{C}$ with real part large enough

$$\left| \frac{k_2 d_2 \, e^{-s\tau}}{s-c_2} \right| < 1 \tag{A.48}$$

and (A.47a) may be expressed as a geometric series

$$\frac{bk_2 x_o \, e^{-s\tau}}{(s-a)((s-c_2) - k_2 d_2 \, e^{-s\tau})}$$

$$= \frac{bk_2 x_o \, e^{-s\tau}}{(s-a)(s-c_2) \left\{ 1 - \frac{k_2 d_2 \, e^{-s\tau}}{(s-c_2)} \right\}}$$

$$= bk_2 x_o \sum_{n=1}^{\infty} \frac{(k_2 d_2)^{n-1} \, e^{-sn\tau}}{(s-a) \; (s-c_2)^n} \tag{A.49}$$

Now, expanding by partial fractions

$$\frac{1}{(s-a)(s-c_2)^n} = \frac{1}{(a-c_2)^n} \frac{1}{(s-a)} + \sum_{i=o}^{n-1} \frac{(-1)^i}{(c_2-a)^{i+1}} \frac{1}{(s-c_2)^{n-i}} \tag{A.50}$$

and adopting the notation

$$\mathcal{L}^{-1} \frac{(k_2 d_2)^{n-1}}{(s-a)(s-c_2)^n} = g_n(t) \tag{A.51}$$

the inverse Laplace transform of (A.50) shows the output for mismatch in b is given by

$$x(t) = e^{at}x_o - bk_2x_o \sum_{n=1}^{\infty} g_n(t-n\tau)\, h(t-n\tau) \tag{A.52a}$$

where

$$g_n(t) = (k_2d_2)^{n-1} \left\{ \frac{e^{at}}{(a-c_2)^n} + \sum_{i=o}^{n-1} g_{n_i}(t) \right\} \tag{A.52b}$$

$$g_{n_i}(t) = \frac{(-1)^i}{(c_2-a)^{i+1}} \frac{t^{n-i-1}}{(n-i-1)!} e^{c_2t} \tag{A.52c}$$

$$c_2 = a - \ell_2b_o \ , \quad k_2 = \ell_2 e^{a\tau} \ , \quad \ell_2 = \frac{a + \sqrt{a^2+b_o^2q/r}}{b_o} \tag{A.52d}$$

and

$$d_2 = b_o - b \tag{A.52e}$$

A.2.6 The Reduction of the Output to Matched Form

When plant and model are matched $b_o = b$ and the parameters of (A.52d) and (A.52e) satisfy

$$d_2 = d = o \ , \quad \ell_2 = \ell \ , \quad k_2 = k \ , \quad c_2 = c \tag{A.53}$$

The terms of (A.52b) containing d_2 are therefore zero

$$g_n(t) = o \qquad n = 2,3,... \tag{A.54}$$

in which case, (A.52a) reduces to

$$x(t) = e^{at}x_o - bkx_o\, g_1(t-\tau)\; h(t-\tau) \tag{A.55}$$

From (A.52b)

$$g_1(t-\tau) = \frac{e^{a(t-\tau)}}{a-c} + g_{1_o}(t-\tau)$$

$$= \frac{e^{a(t-\tau)}}{a-c} + \frac{e^{c(t-\tau)}}{c-a}$$

$$= \frac{e^{a(t-\tau)} - e^{(a-\ell b)(t-\tau)}}{b\ell} \tag{A.56}$$

and substituting (A.56) into (A.55) mismatched output (A.52a) reduces to

$$x(t) = e^{at}x_o - e^{a\tau}x_o\,(e^{a(t-\tau)} - e^{(a-\ell b)(t-\tau)})\; h(t-\tau)$$

$$= e^{at}x_o - (e^{at}x_o - e^{(a-\ell b)(t-\tau)}e^{a\tau}x_o)h(t-\tau) \tag{A.57}$$

An alternative presentation of (A.57) is

$$x(t) = \begin{cases} e^{at}x_o & 0 \leqslant t \leqslant \tau \\ e^{(a-\ell b)(t-\tau)}e^{a\tau}x_o & t \geqslant \tau \end{cases} \tag{A.58}$$

the desired first-order version of matched output (3.29).

A.2.7 The Output over $[0,\tau]$

When $t < \tau$, by the definition of the Heaviside step function

$$h(t-n\tau) = 0 \qquad n = 1,2,... \tag{A.59}$$

and (A.52a) reduces to

$$x(t) = e^{at}x_o \qquad 0 \leqslant t < \tau \tag{A.60}$$

It therefore remains to consider the case of $t = \tau$. In this instance

$$h(t-n\tau) = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,... \end{cases} \tag{A.61}$$

which ensures (A.52a) satisfies

$$x(\tau) = e^{a\tau}x_o - bk_1x_og_1(0) \tag{A.62}$$

From (A.52b) and (A.52c)

$$\begin{aligned} g_1(t) &= \frac{e^{at}}{a-c_2} + g_{1_o}(t) \\ &= \frac{e^{at}}{a-c_2} + \frac{e^{c_2t}}{c_2-a} \end{aligned} \tag{A.63}$$

in which case, when $t = 0$

$$g_1(0) = \frac{1}{a-c_2} + \frac{1}{c_2-a} = 0 \tag{A.64}$$

Substituting (A.64) into (A.62) shows that for the case $t = \tau$ (A.52a) reduces to

$$x(\tau) = e^{a\tau}x_o \tag{A.65}$$

Combining (A.60) and (A.65)

$$x(t) = e^{at}x_o \qquad 0 \leqslant t \leqslant \tau \tag{A.66}$$

the expected form of the mismatched output over $[0,\tau]$.

A.2.8 The Continuity of the Output

As the input in the presence of mismatch in b is continuous for $t \neq \tau$, so is (A.52) the corresponding output. Furthermore, as (A.65) shows (A.52) to be continuous for $t = \tau$, (A.52) is continuous for $t \geq 0$. Continuity of (A.52) could be shown directly following the style adopted in Section A.2.4 for the mismatched input.

A.3 The Form and Properties of the Subplant Output for Mismatch in Delay

A.3.1 The Form of the Output

For mismatch in delay, the Laplace transform of the output is given by

$$\bar{x}(s) = \frac{x_o}{s-a} - \frac{bk_3x_o\, e^{-s\tau}}{(s-a)((s-c) - k_3d_3(s)\, e^{-s\tau})} \tag{A.67a}$$

where

$$c = a - \ell b\ , \quad k_3 = \ell e^{a\tau_o}\ , \quad \ell = \frac{a + \sqrt{a^2+b^2q/r}}{b} \tag{A.67b}$$

and

$$d_3(s) = b(e^{-s(\tau_o-\tau)} - 1) \tag{A.67c}$$

For $s \in \mathbb{C}$ with real part large enough

$$\left| \frac{k_3\ b(e^{-s\tau_o} - e^{-s\tau})}{s-c} \right| < 1 \tag{A.68}$$

and (A.67a) may be expressed as a geometric series

$$\frac{bk_3x_o e^{-s\tau}}{(s-a)\{(s-c)-k_3b(e^{-s\tau_o} - e^{-s\tau})\}}$$

$$= \frac{bk_3x_o\ e^{-s\tau}}{(s-a)(s-c)\left\{1 - \dfrac{k_3b(e^{-s\tau_o} - e^{-s\tau})}{s-c}\right\}}$$

$$= bk_3x_o \sum_{n=1}^{\infty} \frac{(k_3b)^{n-1}\ (e^{-s\tau_o} - e^{-s\tau})^{n-1}\ e^{-s\tau}}{(s-a)(s-c)^n}$$

$$= bk_3x_o \sum_{n=1}^{\infty} \sum_{i=o}^{n-1} (k_3b)^{n-1} \frac{(-1)^{n-i-1}(n-1)!}{i!\ (n-i-1)!} \frac{e^{-s(i\tau_o+(n-i)\tau)}}{(s-a)(s-c)^n} \tag{A.69}$$

Now, expanding by partial fractions

$$\frac{1}{(s-a)(s-c)^n} = \frac{1}{(a-c)^n}\ \frac{1}{(s-a)} + \sum_{j=o}^{n-1} \frac{(-1)^j}{(c-a)^{j+1}}\ \frac{1}{(s-c)^{n-j}} \tag{A.70}$$

and adopting the notation

$$\mathcal{L}^{-1}\ \frac{(k_3b)^{n-1}}{(s-a)(s-c)^n} = g_n(t) \tag{A.71}$$

the inverse Laplace transform of (A.69) shows the output for mismatch in delay is given by

$$x(t) = e^{at}x_o - bk_3x_o \sum_{n=1}^{\infty} \sum_{i=o}^{n-1} m(n,i)g_n(t-\lambda_{n_i})\ h(t-\lambda_{n_i}) \tag{A.72a}$$

where

$$g_n(t) = (k_3b)^{n-1} \left\{ \frac{e^{at}}{(a-c)^n} + \sum_{j=o}^{n-1} g_{n_j}(t) \right\} \tag{A.72b}$$

$$g_{n_j}(t) = \frac{(-1)^j}{(c-a)^{j+1}} + \frac{t^{n-j-1}}{(n-j-1)!}\ e^{ct} \tag{A.72c}$$

$$m(n,i) = \frac{(-1)^{n-i-1}(n-1)!}{i!\ (n-i-1)!}\ ,\quad \lambda_{n_i} = i\tau_o + (n-i)\tau \quad o \leq i \leq n-1 \tag{A.72d}$$

and

$$c = a - \ell b\ ,\quad k_3 = \ell e^{a\tau_o}\ ,\quad \ell = \frac{a + \sqrt{a^2+b^2q/r}}{b} \tag{A.72e}$$

A.3.2 The Reduction of the Output to Matched Form

When plant and model match $\tau_o = \tau$ and the parameters of (A.72d) and (A.72e) satisfy

$$k_3 = k\ ,\quad \lambda_{n_i} = n\tau \quad o \leqslant i \leqslant n-1 \tag{A.73}$$

which ensures (A.72a) becomes

$$x(t) = e^{at}x_o - bkx_o = \sum_{n=1}^{\infty} \sum_{i=o}^{n-1} m(n,i)g_n(t-n\tau)\ h(t-n\tau) \tag{A.74}$$

The sum over i now involves only the terms of $m(n,i)$

$$\sum_{i=o}^{n-1} m(n,i) = \sum_{i=o}^{n-1} \frac{(-1)^{n-i-1}(n-1)!}{i!\ (n-i-1)!} = \begin{cases} 1 & n = 1 \\ o & n = 2,3,\ldots \end{cases} \tag{A.75}$$

and (A.74) reduces to

$$x(t) = e^{at}x_o - bkx_o g_1(t-\tau)\ h(t-\tau) \tag{A.76}$$

From (A.72b) and (A.72c)

$$g_1(t-\tau) = \frac{e^{a(t-\tau)}}{a-c} + g_{1_o}(t-\tau)$$

$$= \frac{e^{a(t-\tau)}}{a-c} + \frac{e^{c(t-\tau)}}{c-a}$$

$$= \frac{e^{a(t-\tau)} - e^{(a-\ell b)(t-\tau)}}{b\ell} \tag{A.77}$$

Substituting (A.77) into (A.76), mismatched output (A.72a) reduces to

$$x(t) = e^{at}x_o - e^{a\tau}x_o\ (e^{a(t-\tau)} - e^{(a-\ell b)(t-\tau)})h(t-\tau)$$

$$= e^{at}x_o - (e^{at}x_o - e^{(a-\ell b)(t-\tau)}\ e^{a\tau}x_o)\ h(t-\tau) \tag{A.78}$$

an alternative presentation of which is

$$x(t) = \begin{cases} e^{at}x_o & 0 \leqslant t \leqslant \tau \\ e^{(a-\ell b)(t-\tau)}\ e^{a\tau}x_o & t \geqslant \tau \end{cases} \tag{A.79}$$

the desired first-order version of matched output (3.29).

A.3.3 The Output over $[0,\tau]$

When $t < \tau$, by definition of the Heaviside step function

$$h(t - \lambda_{n_i}) = 0 \qquad n = 1,2,... \tag{A.80}$$

and (A.72a) reduces to

$$x(t) = e^{at}x_o \qquad 0 \leqslant t < \tau \tag{A.81}$$

It therefore remains to consider the case of $t = \tau$. In this instance

$$h(t - \lambda_{n_i}) = \begin{cases} 1 & n = 1 \\ 0 & n = 2,3,... \end{cases} \tag{A.82}$$

which ensures (A.72a) satisfies

$$x(\tau) = e^{a\tau}x_o - bkx_o\ m(1,0)\ g_1(0) \tag{A.83}$$

From (A.72b), (A.72c) and (A.72d)

$$m(1,0) = 1 \tag{A.84}$$

and

$$g_1(t) = \frac{e^{at}}{a-c} + \frac{e^{ct}}{c-a} \tag{A.85}$$

in which case, when $t = 0$

$$g_1(0) = 0 \tag{A.86}$$

Substituting (A.86) into (A.83) shows that for the case of $t = \tau$ (A.72a) reduces to

$$x(\tau) = e^{a\tau}x_o \tag{A.87}$$

Combining (A.81) and (A.87)

$$x(t) = e^{at}x_o \qquad 0 \leqslant t \leqslant \tau \qquad (A.88)$$

the expected form of the mismatched output over $[0,\tau]$.

A.3.4 The Continuity of the Output

As the input in the presence of mismatch in delay is continuous for $t \neq \tau$, so is (A.72) the corresponding output. Furthermore, as (A.87) shows (A.72) to be continuous for $t = \tau$, (A.72) is continuous for $t \geqslant 0$. Continuity of (A.72) could be shown directly, following the style adopted in Section 4.4.4 for the mismatched input.